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PARTIAL DIFFERENTIAL EQUATIONS.

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PARTIAL DIFFERENTIAL EQUATIONS.

AN ESSAY

TOWARDS AN ENTIRELY

NEW METHOD OF INTEGRATING THEM.

BY

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PREFACE.

THE Author of this Essay has been forewarned that the adoption of the subscript differential notation, which forty years ago was in some degree of favour but afterwards laid aside, will prove a serious hinderance to the acceptableness of the present Work, as it will be new to the generality of mathematical students, and therefore somewhat repellent. He regrets the circumstance, and feels that an explanation and apology are needed; and he finds both in the fact that the notation alluded to has been forced upon him by the necessities of the case; and, moreover, he ventures to think that the reader after going through the Essay will agree with him in this, and be willing on this ground to excuse such an important deviation from established mathematical usage.

The Author denominates his Work an ESSAY, because as a TREATISE it would be very incomplete, there being many portions of the general subject not alluded to in these pages. For information on those portions he must refer the reader to Boole and Gregory and Carmichael, with whose admirable and valuable Works he has been careful to interfere as little as possible. In fact, he has consulted them only for such illustrative examples as fell within the scope of his Essay. He is conscious of having apparently intermeddled with the integration of equations of *one* independent variable; but this has arisen out of the peculiarities of the Author's system, by which every equation, whatever be the number of its independent variables, is reduced to an equivalent equation of only one independent variable. It was therefore impossible to keep the two branches of differential equations apart.

The system of integration here proposed occurred to the mind of the Author a few years ago, but his professional engagements did not then leave him leisure to follow the general idea into its details. And, as his object is merely to render his Method thoroughly intelligible, and not the exhibition of integrals, he has for brevity's sake not always proceeded to the last steps of an integration when he conceived that the method had been made intelligible.

The Author fears a cursory glance at the pages of the Work will have a prejudicial effect; for he is aware that some of them exhibit a formidable and deterrent array of novel symbols; he therefore begs to assure the reader that the various steps of the investigations are all obedient to one general principle, and though in some degree novel, are not really difficult, but on the contrary easy when the eye has become accustomed to the novelties of the notation. And, moreover, he entertains a hope that the results of integrations (many of which are far more general than they were in the shape in which they have appeared in former Treatises) will repay the reader for any extra trouble he may find in pushing through what may at first appear to him a forest of operative symbols. Many equations will be found in this Essay integrated with ease in finite terms, which, as far as the Author is aware, were never integrated in finite terms before. In the last Chapter, for instance, the following highly important equations will be found integrated for the first time in finite terms;

$$\frac{d^2u}{dx\,dy}+cu=0;$$

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+cu=0;$$

and
$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0.$$

Three more important partial differential equations could hardly be instanced, and a finite integral of each one of them is obtained in this Essay without any difficulty.

The Author can scarcely hope that in a work so novel in its methods as this is, and written as this has been under considerable pressure of other engagements, some clerical errors will not have escaped his notice; but he hopes that they will not prove to be many, and that none of them will involve errors of principle. If any are found, he will be thankful if the courteous reader will send notice of them addressed to him at the Publishers.

SHEFFIELD,
October 2, 1871.

ERRATA.

Page 63, line 19, *for* question *read* equation.

,, 69, ,, 18, for f read F.

,, 70, bottom line, for f read F.

AN ESSAY.

CHAPTER I.

GENERAL PRINCIPLES OF THE ESSAY.

IT is known that the integration of differential equations is oftentimes much facilitated by a change of one or more of the independent variables, and sometimes also by changing the dependent variable. The process is however often tedious, and it is frequently far from obvious what ought to be chosen as the new variables, even when it is seen that a change should be made.

The system of integration to be proposed in this Essay is founded on a simple principle of choosing new variables; and the change of variables is effected with great ease. And the most important feature of the system is, that whatever be the number of original independent variables, the work of integration is at once reduced to the use of one independent variable only.

Quasi-constants.

1. If $t, x, y, z \ldots$ be n independent variables, and u the dependent variable, it is understood that $d_t u$, $d_x u$, $d_y u$, $d_z u, \ldots$ are *partial* differential coefficients; by which is meant that in finding the value of $d_t u$, i.e. in operating on u with the symbol d_t, we are to consider $x, y, z, \ldots$ as *constants:* in operating on u with d_x, then $t, y, z, \ldots$ are to be considered as

constants; and so on. It is here proposed to express this briefly by the use of the word *quasi-constant.* Thus in reference to the use of the symbol d_t, we shall denominate $x, y, z, \ldots$ and any functions of them *quasi-constants:* the use of d_x will imply that $t, y, z \ldots$ and their functions are quasi-constants; and so on.

2. If s_t be the symbol of integration corresponding to d_t the symbol of differentiation, then we know that after the operation s_t we must add to complete the integral an arbitrary function of $x, y, z, \ldots$ its quasi-constants: and this principle is general and may be thus stated,—after an operation of integration we are to add to the integral *an arbitrary function of the corresponding quasi-constants;* and all the variables which were considered constants in differentiation are to be considered quasi-constants in integration.

Change of the independent variables.

3. If from the (n) independent variables $t, x, y, z, \ldots$ we wish to pass to a new set $\tau, \xi, \eta, \zeta \ldots$, formulæ for this purpose are ready to hand in the *Differential Calculus.* They are of the following type:

$$(d_\tau u) = d_t u \,.\, (d_\tau t) + d_x u \,.\, (d_\tau x) + d_y u \,.\, (d_\tau y) + \ldots\ldots (1).$$

This is only one of the (n) requisite equations, and the other $(n-1)$ equations may be obtained from this by writing successively in this $\xi, \eta, \zeta, \ldots$ for τ. But it is possible, whenever the forms of $\xi, \eta, \zeta \ldots$ are at our disposal, to assign them such values in terms of $t, x, y, z, \ldots$ as shall render the above equation (1) sufficient by itself for our purpose; so that by a proper choice of values for $\xi, \eta, \zeta, \ldots$ any partial differential equation of n variables may be reduced to an equivalent equation of *one variable* and $(n-1)$ *quasi-constants.* This is the principle on which this Essay is founded. The method of assigning proper forms for the quasi-constants $\xi, \eta, \zeta, \ldots$ in terms of $t, x, y, z, \ldots$ the original independent variables will be best learned from the examples which will be given.

Distinctive Symbols of operation for different sets of independent variables.

4. In the formula (1) certain terms are enclosed in brackets; this is done to indicate that before the differential operations indicated in them are performed, $t, x, y \ldots$ must be expressed in terms of the new variables; or, that the operations are to be performed on the supposition that u is so expressed. In brief, the distinction of brackets and no brackets in equation (1) refers to the fact that there are operations to be performed in that equation in reference to two sets of independent variables. Had we to write down an equation in which differential and integral operations would have to be performed in reference to three or more sets of independent variables, we should have to employ two or more kinds of distinctive brackets to indicate that fact. Now it is proposed in this Essay to indicate this fact by employing distinctive differential and integral symbols of operation, and so to abolish the distinction by brackets. Thus, when $\tau, \xi, \eta, \zeta \ldots$ is the set of independent variables referred to in a differential or integral operation, we shall use D and S as the symbols of operation; if $\tau', \xi', \eta', \zeta', \ldots$ be the set of independent variables, we shall indicate that fact by using Δ and Σ as the symbols of operation; and so on for any number of sets. Under this system equation (1) of Art. 3 will be written without brackets thus,—

$$D_\tau u = d_t u \,.\, D_\tau t + d_x u \,.\, D_\tau x + d_y u \,.\, D_\tau y + \ldots\ldots\ldots \quad (1).$$

5. In speaking about different sets of independent variables, it is to be noticed that it is not necessary that *every* individual member of one set should be different from every

individual member of another set. If *some*, even if *one*, of one set be different from all of another set this is sufficient to establish them as different sets, and to entitle them to the use of distinctive differential and integral symbols of operation.

6. In fixing the values of a new set of independent variables in terms of the original set, it is of importance to take care that each member of the new set be really an *independent* variable; i.e. that it be such as cannot be expressed in terms of the other members of its own set. Unless this condition of independence be observed, we shall be working under an erroneous hypothesis.

7. Under the authority of Art. 5 we are at liberty to assume, whenever it can be done usefully, that one of the variables of each new set shall be identical with one of the variables of the original set,—for example, we may assume $\tau = t$, in which case equation (1) of Art. 4 becomes

$$D_t u = d_t u + d_x u . D_t x + d_y u . D_t y + \dots\dots\dots\dots \text{(1)}.$$

But when we use this equation, which will always be known from the occurrence of the symbol D_t or S_t, we must remember that there remain only $(n-1)$ new independent variables (the quasi-constants) of the set to be determined, one of them (t) being already fixed upon in using this equation.

8. In determining what shall be the forms of the constituent members of a new set of variables, we are at liberty to introduce n *arbitrary* hypotheses if we use equation (1) of Art. 4; but only $(n-1)$ arbitrary hypotheses if we employ equation (1) of the last Article. The arbitrariness of these hypotheses is however overruled by the necessity that the resulting members of a set of variables determined from them must be independent (Art. 6).

CHAPTER II.

DETERMINATION OF QUASI-CONSTANTS. EQUATIONS OF THE FIRST ORDER.

Ex. 1. To integrate $d_t u + a d_x u = bu$.

Let us change the independent variables to t, ξ. Then the formula of Art. 7 becomes for this case, which is one of two independent variables,

$$D_t u = d_t u + D_t x . d_x u \text{ .... } (1).$$

We are at liberty to make *one* hypothesis (Art. 8) for the determination of the value of ξ; let us assume $D_t x = a$. We make this choice, because by this assumption the right-hand member of the formula (1) will be rendered identical with the left-hand member of the proposed equation; so that we shall have $D_t u = bu$;

$$\therefore\ D_t x = a, \text{ and } D_t u = bu,$$

and these two equations, taken simultaneously, are together exactly equivalent to the proposed equation. Integrating them we have (see Art. 2)

$$x = at + \xi, \text{ and } \epsilon^{-bt} u = F(\xi).$$

The former gives the value of ξ; and this value being written in the latter, we have the following as a relation that must always exist between u and the independent variables t, x,

$$u = \epsilon^{bt} F(x - at).$$

This is therefore the required integral.

9. Remark. According to Art. (2) we ought to have written $x = at + f(\xi)$ instead of $x = at + \xi$, to complete the integration of $D_t x = a$: but had we done so the final integral would not have been affected thereby: and we were moreover at liberty to take either $x - at$, or any function of $x - at$ as the value of ξ. Under this license no additional generality would be gained by the use of $f\xi$ instead of ξ. We shall always make use of this liberty by adding either ξ, or a function of ξ, after *one* of the integrations in any proposed problem.

Ex. 2. To integrate $d_t u + a d_x u + b d_y u = cu$.

In this case, which is one of three independent variables, we have to change the variables to t, ξ, η; and the formula of Art. 7 becomes

$$D_t u = d_t u + D_t x . d_x u + D_t y . d_y u.$$

We are here at liberty to make two hypotheses for the determination of the quasi-constants ξ and η. We therefore assume $D_t x = a$, $D_t y = b$; which gives $D_t u = cu$. These three equations, when taken simultaneously, are equivalent to the proposed equation.

We shall employ the first and second for fixing the values of ξ and η;

$$\therefore \; x = at + \xi, \; y = bt + \eta, \text{ and } \epsilon^{-ct} u = F(\xi, \eta).$$

And by substituting in the last of these the values of ξ and η found from the other two we find, as the required integral,

$$u = \epsilon^{ct} F(x - at, y - bt).$$

Ex. 3. To integrate $d_t u + \frac{t}{x} \cdot d_x u = \frac{tu}{x^2}$.

Comparing this, which is a case of two independent variables, with $D_t u = d_t u + D_t x . d_x u$, we find the two following equations which when taken simultaneously are equivalent to the proposed equation;

$$D_t x = \frac{t}{x}, \text{ and } D_t u = \frac{tu}{x^2}.$$

The former by integration gives $x^2 = t^2 + \xi$, which fixes the value of ξ: and the two equations combined give

$$xD_t u - uD_t x = 0,$$

which being integrated gives $\frac{u}{x} = F(\xi)$;

$$\therefore u = xF(x^2 - t^2).$$

Ex. 4. To integrate $d_t u + \frac{u}{t} \cdot d_x u = \frac{a^2 x}{t}$.

Comparing this with the formula $D_t u = d_t u + D_t x . d_x u$, we have

$$D_t x = \frac{u}{t}, \text{ and } D_t u = \frac{a^2 x}{t};$$

$$\therefore uD_t u - a^2 x D_t x = 0;$$

$$\therefore u^2 - a^2 x^2 = \xi.$$

Also $$D_t x = \frac{u}{t} = \frac{\sqrt{a^2 x^2 + \xi}}{t};$$

$$\therefore \log (ax + \sqrt{a^2 x^2 + \xi}) = a \log t + \log F(\xi);$$

$$\therefore ax + u = t^a . F(u^2 - a^2 x^2).$$

Ex. 5. To integrate $d_t u + \frac{x}{t} . d_x u + \frac{a}{t} . d_y u = \frac{nu}{t}$.

Comparing this, which is a case of three independent variables, with

$$D_t u = d_t u + D_t x . d_x u + D_t y . d_y u,$$

we have

$$D_t x = \frac{x}{t}, \quad D_t y = \frac{a}{t}, \text{ and } D_t u = \frac{nu}{t}.$$

These equations being integrated give

$$x = t\xi, \quad \epsilon^{\frac{y}{a}} = t\eta, \text{ and } t^{-n} u = F(\xi, \eta);$$

$$\therefore\ u = t^n F\left(\frac{x}{t}, \ \frac{1}{t} . \epsilon^{\frac{y}{a}}\right).$$

Ex. 6. To integrate $d_x V . d_t u - d_t V . d_x u = 0$.

Divide by $d_x V$, and compare with

$$D_t u = d_t u + D_t x . d_x u;$$

$$\therefore\ D_t x = -\frac{d_t V}{d_x V}, \text{ and } D_t u = 0.$$

The latter gives by integration $u = \xi$, and the former may be written in the form $0 = d_t V + D_t x . d_x V$ which is $= D_t V$. And hence, by integration,

$$V = F(\xi) = F(u).$$

Hence, in an equation of the form proposed, the dependent variables u, V are necessarily functions of one another. We may write the integral under the form

$$f(u, V) = 0.$$

Ex. 7. To integrate $d_t u - \frac{x}{t} . d_x u + \frac{y}{t} . d_y u = \frac{y^2}{t^2 x}$.

Comparing this equation with

$$D_tu = d_tu + D_tx \,.\, d_xu + D_ty \,.\, d_yu,$$

we have

$$D_tx = -\frac{x}{t}, \quad D_ty = \frac{y}{t}, \text{ and } D_tu = \frac{y^2}{t^2x}.$$

The first two give $tx = \xi$ and $y = t\eta$, and these being used in the third to eliminate x and y, we have

$$D_tu = \frac{\eta^2}{\xi} \,.\, t, \quad (\xi, \eta \text{ are quasi-constants});$$

$$\therefore\ u = \frac{\eta^2}{\xi} \,.\, \frac{t^2}{2} + F(\xi, \eta)$$

$$= \frac{y^2}{2tx} + F\left(tx, \frac{y}{t}\right).$$

Ex. 8. To integrate $d_tu - \frac{x}{t} \,.\, d_xu + \frac{y}{x} d_yu = \frac{y^2}{tx}$.

In this example,

$$D_tx = -\frac{x}{t}, \quad D_ty = \frac{y}{x}, \quad \text{and } D_tu = \frac{y^2}{tx};$$

$$\therefore\ \xi = tx, \quad \eta = y\epsilon^{-\frac{t^2}{2\xi}}, \text{ and } D_tu = \frac{\eta^2}{\xi} \,.\, \epsilon^{\frac{t^2}{\xi}};$$

$$\therefore\ u = \frac{\eta^2}{\xi} \,.\, S_t\epsilon^{\frac{t^2}{\xi}} + F(\xi, \eta)$$

$$= \frac{y^2}{tx} \,.\, \epsilon^{-\frac{t}{x}} S_t\epsilon^{\frac{t^2}{\xi}} + F(tx, y\epsilon^{-\frac{t}{2x}}).$$

Ex. 9. To integrate $d_tu + ad_xu + \dfrac{bu}{ht + kx} = 0$.

Comparing this with the usual formula we have $D_tx = a$, and $(ht + kx)\, D_tu = -bu$. Hence

$$x = at + \xi; \text{ and } D_t\,(ht + kx) = h + ka.$$

It still remains to integrate

$$(ht + kx) D_t u + bu = 0.$$

Multiply by $(ht + kx)^{m-1}$, m being at present an unknown constant;

$$\therefore \ (ht + kx)^m D_t u + b (ht + kx)^{m-1} u = 0.$$

We see at a glance that this will be an exact differential, if m be such that $b = mD_t (ht + kx)$, that is, if

$$b = m (h + ka);$$

$$\therefore \ (ht + kx)^m u = F(\xi) = F(x - at).$$

Ex. 10. To integrate $d_t u + \frac{x}{t} . d_x u = \frac{au}{ht + kx}$.

In this example,

$$D_t x = \frac{x}{t}, \quad \text{and } D_t u = \frac{au}{ht + kx};$$

$$\therefore \ x = \xi t, \quad \text{and } (ht + kx) D_t u = au;$$

$$\therefore \ tD_t u = \frac{au}{h + k\xi} \ (\xi \text{ is a quasi-constant});$$

$$\therefore \ u = t^{\frac{a}{h + k\xi}} . F(\xi)$$

$$= t^{\frac{at}{ht + kx}} . F\left(\frac{x}{t}\right).$$

Ex. 11. To integrate

$$d_t u + ad_x u + bd_y u = c, \quad \text{and} \quad d_t u + a' d_x u + b' d_y u = c'$$

simultaneously.

Proceeding with the former of these as before, we have

$$\therefore \ D_t x = a, \ D_t y = b, \ D_t u = c;$$

which being integrated give

$$x = at + \xi,\ y = bt + \eta, \text{ and } u = ct + F(\xi, \eta)$$
$$= ct + F(x - at,\ y - bt).$$

Now substitute the value of u, given in the last equation, in the second of the simultaneous equations, using F for $F(x - at, y - bt)$ for brevity,

$$\therefore (a - a')\, d_\xi F + (b - b')\, d_\eta F = c - c'.$$

This is an ordinary differential equation of the first order with constant coefficients, and being integrated as in Examples preceding gives

$$F = f\{(b - b')\,\xi + (a' - a)\,\eta\} + \frac{c\xi}{a - a'} - \frac{c'\eta}{b - b'},$$

$$\therefore\ u = \left(\frac{a'c}{a' - a} + \frac{bc'}{b' - b}\right) t + f\{(b - b')\,x + (a' - a)\,y + (ab' - ba')\,t\}.$$

10. In equations of the first order, if the coefficients be of a symmetrical form, it will sometimes be found to simplify the process of integration if we use the formula of Art. 4, instead of that of Art. 7.

Ex. 12. To integrate $(x + t)\, d_t u + (x - t)\, d_x u = au$.

Comparing this with

$$D_\tau u = d_t u \,.\, D_\tau t + d_x u \,.\, D_\tau x,$$

we have the following equations,

$$D_\tau u = au, \quad D_\tau t = x + t, \quad D_\tau x = x - t.$$

From the last two we obtain

$$tD_\tau t + xD_\tau x = t^2 + x^2,$$

and

$$xD_\tau t - tD_\tau x = t^2 + x^2;$$

$$\therefore\ \log\sqrt{t^2+x^2}=\tau+\phi_1\xi,\ \text{and}\ \tan^{-1}\frac{t}{x}=\tau+\phi_2\xi.$$

Also, the integration of the first gives

$$\log u = a\tau+\phi_3\xi.$$

Between these three equations we have to eliminate τ and ξ;

$$\therefore\ \log u - a\log\sqrt{t^2+x^2}=\phi_3\xi - a\phi_1\xi,$$

and
$$\tan^{-1}\frac{t}{x}-\log\sqrt{t^2+x^2}=\phi_2\xi-\phi_1\xi.$$

Now the right-hand members of these equations being arbitrary functions of ξ, the left-hand members must also be arbitrary functions of each other;

$$\therefore\ u=(t^2+x^2)^{\frac{a}{2}}F\left(\tan^{-1}\frac{t}{x}-\log\sqrt{t^2+x^2}\right).$$

Ex. 13. To integrate

$$(bu-cx)\,d_tu+(ct-au)\,d_xu=ax-bt.$$

Comparing this with the formula of Art. 4, we have

$$D_\tau t=bu-cx,\ D_\tau x=ct-au,\ \text{and}\ D_\tau u=ax-bt;$$

$$\therefore\ aD_\tau t+bD_\tau x+cD_\tau u=0,\ \text{and}\ tD_\tau t+xD_\tau x+uD_\tau u=0.$$

$$\therefore\ at+bx+cu=\xi,\ \text{and}\ t^2+x^2+u^2=F(\xi);$$

$$\therefore\ t^2+x^2+u^2=F(at+bx+cu),$$

which is the integral required.

Ex. 14. To integrate

$$(x+y+u)\,d_tu+(y+t+u)\,d_xu+(t+x+u)\,d_yu=t+x+y.$$

$$\begin{aligned} \therefore\ D_\tau t &= x+y+u,\\ D_\tau x &= y+t+u,\\ D_\tau y &= t+x+u,\\ \text{and }\ D_\tau u &= t+x+y; \end{aligned}$$

$$\therefore\ \frac{D_\tau t - D_\tau x}{t-x} = \frac{D_\tau x - D_\tau y}{x-y} = \frac{D_\tau y - D_\tau u}{y-u} = -1,$$

$$\text{and } D_\tau (t+x+y+u) = 3\,(t+x+y+u).$$

Integrating these equations, and writing Q for $t+x+y+u$ for brevity, we have

$$t-x=\epsilon^{-\tau}F_1(\xi,\eta),\quad x-y=\epsilon^{-\tau}F_2(\xi,\eta),\quad y-u=\epsilon^{-\tau}F_3(\xi,\eta),$$

$$\text{and } Q=\epsilon^{3\tau}F_4(\xi,\eta).$$

Eliminating τ from these integrals, we find that

$$(t-x)\sqrt[3]{Q},\quad (x-y)\sqrt[3]{Q},\quad (y-u)\sqrt[3]{Q}$$

are (each of them) arbitrary functions of ξ, η; consequently each of them is an arbitrary function of the other two.

$$\therefore\ 0=F\{(t-x)\sqrt[3]{Q},\ (x-y)\sqrt[3]{Q},\ (y-u)\sqrt[3]{Q}\}.$$

Ex. 15. To integrate $t^2 d_t u + x^2 d_x u = atx$.

Comparing this with the formula of Art. 4, we have

$$D_\tau u = atx,\quad D_\tau t = t^2,\quad D_\tau x = x^2.$$

From the last two of these we find $x^2 D_\tau t = t^2 D_\tau x$,

$$\therefore\ \frac{1}{t}=\frac{1}{x}+\xi;$$

$$\therefore\ D_\tau u = ax^2\,.\,\frac{t}{x} = \frac{aD_\tau x}{1+\xi x};$$

$$\therefore\ u = \frac{a}{\xi}\,.\log(1+\xi x) + F\xi$$

$$= \frac{atx}{x-t}\,.\log\frac{x}{t} + F\left(\frac{1}{t}-\frac{1}{x}\right),$$

which is the required integral.

Ex. 16. To integrate

$$(x^3t - 2t^4)\,d_tu + (2x^4 - t^3x)\,d_xu = 9\,(t^3 - x^3)\,u.$$

Comparing this with the formula of Art. 4, we have

$$D_\tau u = 9\,(t^3 - x^3)\,u,\quad D_\tau t = x^3t - 2t^4,\quad D_\tau x = 2x^4 - t^3x.$$

$$\therefore\ \frac{D_\tau u}{3u} + \frac{D_\tau t}{t} + \frac{D_\tau x}{x} = 0,$$

$$\text{and}\quad \frac{xD_\tau t - 2tD_\tau x}{x^3} + \frac{tD_\tau x - 2xD_\tau t}{t^3} = 0;$$

$$\therefore\ t^3x^3u = F(\xi),\ \text{and}\ \frac{t}{x^2} + \frac{x}{t^2} = \xi;$$

$$\therefore\ t^3x^3u = F\left(\frac{t}{x^2} + \frac{x}{t^2}\right)$$

which is the integral required.

CHAPTER III.

DETERMINATION OF QUASI-CONSTANTS. EQUATIONS OF THE SECOND AND HIGHER ORDERS.

11. SINCE $D_tu = d_tu + D_tx \,.\, d_xu + D_ty \,.\, d_yu + \ldots$ it follows that the single operation denoted by D_t is equivalent to the compound system of operations denoted by

$$d_t + D_tx \,.\, d_x + D_ty \,.\, d_y + \ldots$$

Hence if D_tx, D_ty, ... are either constants, or quasi-constants, any function of the one operation D_t will be equivalent to the same function of the compound system

$$d_t + D_tx \,.\, d_x + D_ty \,.\, d_y + \ldots$$

$$\therefore\ F(D_t) = F(d_t + D_tx \,.\, d_x + D_ty \,.\, d_y + \ldots) \ldots\ldots\ldots (1).$$

Ex. 1. To integrate

$$d_t^2u + 2ad_t d_xu + a^2d_x^2u + b\,(d_tu + ad_xu) + cu = 0.$$

Taking the terms of the second order, we have

$$d_t^2 + 2ad_td_x + a^2d_x^2 = (d_t + ad_x)^2.$$

We therefore assume $D_tu = d_tu + ad_xu$, which compared with the general formula gives

$$D_t x = a; \text{ and } \therefore D_t^2 u + bD_t u + cu = 0;$$

$$\therefore x = at + \xi, \text{ and } u = \epsilon^{mt} F(\xi) + \epsilon^{nt} f(\xi)$$
$$= \epsilon^{mt} F(x - at) + \epsilon^{nt} f(x - at),$$

m and n being the two roots of $m^2 + bm + c = 0$.

Ex. 2. To integrate

$$d_t^2 u - a^2 d_x^2 u + b(d_t u + ad_x u) = 0.$$

Since $d_t^2 - a^2 d_x^2 = (d_t + ad_x)(d_t - ad_x)$, we shall ha (here for the first time) to use *two* new sets of independe variables, corresponding to the two equations,

$$D_t u = d_t u + ad_x u, \text{ and } \Delta_t u = d_t u - ad_x u.$$

These give $D_t x = a$, and $\Delta_t x = -a$; and since the syn bols D_t, Δ_t are respectively equivalent to the compoun symbols $d_t + ad_x$, $d_t - ad_x$, the proposed equation becomes

$$D_t \Delta_t u + bD_t u = 0.$$

Hence the following system of three simultaneous equ tions is equivalent to the proposed equation,

$$D_t x = a, \ \Delta_t x = -a, \text{ and } D_t \Delta_t u + bD_t u = 0 \ldots\ldots (1).$$

The first two give by integration,

$$x = at + \xi, \quad x = -at + \xi';$$

and the third being integrated with S_t gives

$$\Delta_t u + bu = F(\xi) = F(x - at) \ldots\ldots\ldots\ldots\ldots (2).$$

But the symbols D_t and Δ_t are commutative because the equivalents $d_t + ad_x$, $d_t - ad_x$ are so; consequently the equ tion (1) may be written in the form

$$\Delta_t D_t u + bD_t u = 0,$$

which being integrated with Σ_t gives

$$D_t u = \epsilon^{-bt} f(\xi') = \epsilon^{-bt} f(x + at).$$

Adding this equation to equation (2), and writing u for the sum of the left-hand members, we obtain finally

$$u = F(x - at) + \epsilon^{-bt} f(x + at),$$

for the integral of the proposed equation.

12. The following is the reason why we may write u for $D_t u + \Delta_t u + bu$.

Since the proposed equation is represented by

$$D_t \Delta_t u + bD_t u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

differentiate this equation separately with D_t and Δ_t; and also multiply it by b; the results are (since D_t and Δ_t are commutative)

$$D_t \Delta_t \,.\, D_t u + bD_t \,.\, D_t u = 0,$$

$$D_t \Delta_t \,.\, \Delta_t u + bD_t \,.\, \Delta_t u = 0,$$

$$D_t \Delta_t \,.\, bu + bD_t \,.\, bu = 0;$$

$$\therefore\ D_t \Delta_t \,.\, (D_t u + \Delta_t u + bu) + bD_t \,.\, (D_t u + \Delta_t u + bu) = 0.$$

This equation is of the same form as (1), and we learn from it that $D_t u + \Delta_t u + bu$ will satisfy the proposed equation quite as well as u will do so. If therefore $D_t u + \Delta_t u + bu$ contain the requisite number of arbitrary functions, it is a complete integral of the proposed equation, and being so we may write u for it.

We shall find this principle of great use hereafter; but it must be noticed that it only applies to cases of linear equations with constant or quasi-constant coefficients and when the symbols of operation are commutative.

13. It is well known that the integral of any linear differential equation of the form $F(d_t, d_x, d_y, \ldots)u = W$, where W is a function of t, x, y, ..., consists of two independent parts; one of them independent of W is called the absolute part of the integral; the other is dependent on W. These two parts being independent of each other, may be found separately (should it be advantageous to do so); and their sum will constitute the complete integral. The absolute part corresponds to the general supposition, $W = 0$.

As an example of this principle, let it be required to integrate the equation $d_t^2u - a^2d_x^2u + b(d_tu + ad_xu) = W$.

We first find, as in the preceding example, the absolute integral, *i.e.* that part of the complete integral which corresponds to $W = 0$. This has already been found to be

$$u = \epsilon^{-bt}f(x + at) + F(x - at).$$

To find the part that depends on W we have the equation

$$D_t\Delta_tu + bD_tu = W, \text{ or } \Delta_tD_tu + bD_tu = W.$$

Either of these (whichever may be judged most convenient) will furnish the required result. For integrating the former with S_t, or the latter with Σ_t, we have

$$\Delta_tu + bu = S_tW, \text{ or } D_tu = \epsilon^{-bt}\Sigma_t\epsilon^{bt}W.$$

Integrate, now, the former with Σ_t, or the latter with S_t;

$$\therefore\ u = \epsilon^{-bt}\Sigma_t\epsilon^{bt}S_tW, \text{ or } u = S_t\epsilon^{-bt}\Sigma_t\epsilon^{bt}W.$$

Adding to either of these the absolute part of the integral, we have for the complete result

$$u = \epsilon^{-bt}\Sigma_t\epsilon^{bt}S_tW + \epsilon^{-bt}f(x + at) + F(x - at),$$

$$\text{or } u = S_t\epsilon^{-bt}\Sigma_t\epsilon^{bt}W + \epsilon^{-bt}f(x + at) + F(x - at).$$

14. From the results just obtained we perceive that the symbols $\epsilon^{-bt}\Sigma_t\epsilon^{bt}$ and S_t are commutative in the preceding example. It will be found always necessary to notice whether the symbols of operation in any proposed example are commutative, for the commutativeness of the symbols of operation is a property of immense use in facilitating the integration of equations; and in some instances failure arises solely out of the want of commutativeness.

15. In seeking the absolute portion of an integral, the following property will be found of great use. It is applicable in all linear equations with constant coefficients; and its proof depends on the property proved in Art. 12.

If $d_t^m D_t^n \Delta_t^p u = 0$, and the symbols d_t, D_t, Δ_t are known to be commutative, then $u = s_t^m O + S_t^n O + \Sigma_t^p O$.

For since $d_t^m D_t^n \Delta_t^p u = 0$ is a linear equation with constant coefficients, operate on it separately with s_t^m, S_t^n, Σ_t^p; the results are

$$\therefore\ D_t^n \Delta_t^p u = s_t^m O, \quad d_t^m \Delta_t^p u = S_t^n O, \quad d_t^m D_t^n u = \Sigma_t^p O.$$

Add these equations together, writing u for the sum of their left-hand members;

$$\therefore\ u = s_t^m O + S_t^n O + \Sigma_t^p O.$$

Now this value of u contains $m+n+p$ arbitrary functions, viz. m corresponding to the symbol s_t^m, n to the symbol S_t^n, and p to the symbol Σ_t^p; and consequently it is the complete integral of the proposed equation, which is a differential equation of the $(m+n+p)$th order.

It will have been noticed in the above proof, that the commutativeness of the symbols d^t, D_t, Δ_t has been assumed; and this assumption was made on the ground that the proposed equation $d_t^m D_t^n \Delta_t^p u = 0$ represents a linear equation with constant coefficients.

16. In a similar way it may be shewn that if F, G, H be any symbols whatever of integration,

$$F.G.H.0 = F.0 + G.0 + H.0,$$

provided the proper number of arbitrary functions be furnished by the right-hand member, and F, G, H be commutative.

Ex. 3. To integrate the equation

$$d_t^2 u + (a+b)\, d_t d_x u + ab d_x^2 u = W.$$

The left-hand member being equal to $(d_t + ad_x)\,(d_t + bd_x)\,u$, we have $D_t = d_t + ad_x$, $\Delta_t = d_t + bd_x$; and therefore $\xi = x - at$, $\xi' = x - bt$; and the proposed equation may be written

$$D_t \Delta_t u = W + 0;$$

$$\begin{aligned} \therefore\ u &= S_t \Sigma_t W + S_t 0 + \Sigma_t 0, \quad \text{(Art. 15)} \\ &= S_t \Sigma_t W + F\xi + f\xi' \\ &= S_t \Sigma_t W + F(x - at) + f(x - bt). \end{aligned}$$

Ex. 4. To integrate $d_t^2 u - a^2 d_x^2 u = W$.

Since $d_t^2 u - a^2 d_x^2 u = (d_t + ad_x)\,(d_t - ad_x)\,u$, we have $D_t = d_t + ad_x$, $\Delta_t = d_t - ad_x$; and therefore $\xi = x - at$, $\xi' = x + at$;

$$\therefore\ D_t \Delta_t u = W + 0;$$

$$\therefore\ u = S_t \Sigma_t W + F(x - at) + f(x + at).$$

Ex. 5. To integrate

$$d_t^2 u = a^2 d_x^2 u + 2ab d_x d_y u + b^2 d_y^2 u.$$

The right-hand member is equivalent to $(ad_x + bd_y)^2 u$; hence the equation is equivalent to

$$\begin{aligned} 0 &= d_t^2 u - (ad_x + bd_y)^2 u \\ &= (d_t + ad_x + bd_y)(d_t - ad_x - bd_y)\, u. \end{aligned}$$

Consequently $D_t = d_t + ad_x + bd_y$, $\Delta_t = d_t - ad_x - bd_y$; whence $\xi = x - at$, $\eta = y - bt$; $\xi' = x + at$, $\eta' = x + bt$; and the proposed equation becomes

$$\begin{aligned} &D_t \Delta_t u = 0; \\ &\therefore\ u = S_t O + \Sigma_t O \quad \text{(Art. 15)}, \\ &\qquad = F(\xi, \eta) + f(\xi', \eta') \\ &\qquad = F(x - at, y - bt) + f(x + at, y + bt). \end{aligned}$$

Ex. 6. To integrate

$$\begin{aligned} d_t^2 u + (a + b)\, d_t d_x u + ab d_x^2 u + A\,(d_t u + a d_x u) \\ + B\,(d_t u + b d_x u) + ABu = W. \end{aligned}$$

In this example $D_t = d_t + ad_x$, $\Delta_t = d_t + bd_x$; hence $\xi = x - at$, $\xi' = x - bt$; and the proposed equation becomes

$$\begin{aligned} &D_t \Delta_t u + A D_t u + B \Delta_t u + ABu = W; \\ &\therefore\ (D_t + B)(\Delta_t + A)\, u = W; \\ &\therefore\ \epsilon^{-Bt} D_t \epsilon^{Bt} \,.\, \epsilon^{-At} \Delta_t \epsilon^{At} u = W \ldots\ldots\ldots (1). \end{aligned}$$

The absolute part of the required integral is (see Art. 16)

$$\begin{aligned} u &= \epsilon^{-Bt} S_t \epsilon^{Bt} O + \epsilon^{-At} \Sigma_t \epsilon^{At} O \\ &= \epsilon^{-Bt} F\xi + \epsilon^{-At} f(\xi') \\ &= \epsilon^{-Bt} F(x - at) + \epsilon^{-At} f(x - bt). \end{aligned}$$

The part dependent on W may be determined from equation (1), which gives

$$u = \epsilon^{-At} \Sigma_t \epsilon^{(A-B)t} S_t \epsilon^{Bt} W.$$

Hence the complete integral of the proposed equation is

$$u = \epsilon^{-At}\Sigma_t \epsilon^{(A-B)t} S_t \epsilon^{Bt} W + \epsilon^{-Bt} F(x - at) + \epsilon^{-At} f(x - bt).$$

17. It may be necessary to shew that $\epsilon^{-at} s_t^n \epsilon^{at}$ is the symbol of integration corresponding to the symbol of differentiation, $\epsilon^{-at} d_t^n \epsilon^{at}$; *i.e.* if $\epsilon^{-at} d_t^n \epsilon^{at} u = v$, then $u = \epsilon^{-at} s_t^n \epsilon^{at} v$. This may be proved as follows.

Since $\epsilon^{-at} d_t^n \epsilon^{at} u = v,$

$$\therefore\ d_t^n \epsilon^{at} u = \epsilon^{at} v;$$

$$\therefore\ \epsilon^{at} u = s_t^n \epsilon^{at} v;$$

$$\therefore\ u = \epsilon^{-at} s_t^n \epsilon^{at} v.$$

Similar formulæ will of course be true for the symbols D_t and S_t; for Δ_t and Σ_t; &c.

Ex. 7. To integrate $d_t^2 u + 3d_t d_x u + 2d_x^2 u = t + x.$

Since $d_t^2 + 3d_t d_x + 2d_x^2 = (d_t + d_x)(d_t + 2d_x)$, we have

$$D_t u = d_t u + d_x u, \quad \Delta_t u = d_t u + 2d_x u;$$

$$\therefore\ D_t x = 1,\ \Delta_t x = 2, \text{ and } x = t + \xi,\ x = 2t + \xi';$$

$$\therefore\ D_t \Delta_t u = t + x.$$

The absolute part of the integral is $S_t \Sigma_t O$, which is equal to $S_t O + \Sigma_t O = F(\xi) + f(\xi') = F(x - t) + f(x - 2t)$.

The part of the integral dependent on W or $t + x$ is found as follows: since $D_t(t + x) = 2$, and $\Delta_t(t + x) = 3$,

$$u = \Sigma_t S_t (t + x) = \Sigma_t . \frac{1}{4}(t + x)^2$$

$$= \frac{1}{36}(t + x)^3.$$

Hence the complete integral, found by adding to this the absolute part, is

$$u = \frac{1}{36} \cdot (t + x)^3 + F(x - t) + f(x - 2t).$$

18. The integration of $t + x$ in the preceding example suggests the necessity of the following general explanation. If in any example we know that $D_t x = a$ and $\Delta_t x = b$; and if we have to integrate such an expression as $f(ht + kx)$, we must remember that

as $\qquad D_t(ht + kx) = h + ka,$ and $\Delta_t(ht + kx) = h + kb,$

$$\therefore\ S_t^m f(ht + kx) = \frac{f_m(ht + kx)}{(h + ka)^m};$$

and $$\therefore\ \Sigma_t^n S_t^m f(ht + kx) = \frac{f_{m+n}(ht + kx)}{(h + ka)^m (h + kb)^n}.$$

Consequently, had the integral of the equation

$$d_t^2 u + (a + b)\, d_t d_x u + ab\, d_x^2 u = (ht + kx)^n$$

been required, we should have found

$$u = \frac{(ht + kx)^{n+2}}{(n + 1)(n + 2)(h + ka)(h + kb)} + F(x - at) + f(x - bt).$$

Ex. 8. To integrate $(d_t - 2d_x)(d_t + d_y)(d_x - 3d_y)\, u = 0.$

Here we have

$$D_t u = d_t u - 2d_x u,\quad \Delta_t u = d_t u + d_y u,\quad \delta_x u = d_x u - 3d_y u;$$

consequently $D_t x = -2$; $\Delta_t y = 1$, $\delta_x y = -3$;

and $$x = -2t + \xi,\quad y = t + \xi',\quad y = -3x + \xi'';$$

and $$D_t \Delta_t \delta_x u = 0;$$

$$\begin{aligned} \therefore\ u &= S_t 0 + \Sigma_t 0 + \sigma_x 0 \\ &= F(\xi, y) + f(\xi', x) + \phi(\xi'', t) \\ &= F(x + 2t, y) + f(y - t, x) + \phi(y + 3x, t). \end{aligned}$$

CHAPTER IV.

THE TREATMENT OF QUASI-CONSTANTS.

19. WE propose in this Chapter to develope more fully the doctrine of quasi-constants, and we think that it will be seen that by a right use of them a very great extension of integration-power is effected; and that this increase of power is almost wholly the consequence of the use of symbols of operation instead of brackets (Art. 4). From what has been said in Arts. 1 and 2, about the nature of quasi-constants, it will be understood that when we meet with such an equation as $D_t u = d_t u + \xi d_x u + \eta d_y u + \zeta d_z u + \dots$ we may treat it as an equation with constant coefficients. Now, on comparing it with the general formula of Art. 7, we find

$$D_t x = \xi, \quad D_t y = \eta, \quad D_t z = \zeta, \text{ \&c. } \dots\dots\dots\dots (1),$$

and these by integration give

$$x = \xi t + \phi_1 \xi, \quad y = \eta t + \phi_2 \eta, \quad z = \zeta t + \phi_3 \zeta, \text{\&c.} \dots\dots (2).$$

We might give to these integrals an apparently still more general form by writing $\phi_1(\xi, \eta, \zeta \dots)$, $\phi_2(\xi, \eta, \zeta \dots)$,

$$\phi_3(\xi, \eta, \zeta \dots), \dots \text{ for } \phi_1\xi, \phi_2\eta, \phi_3\zeta, \dots$$

The forms above given are in general better suited for our purpose; and our conclusion from them is this, that if the independent variables of a new set are such that each one (as ξ) will satisfy an equation of the typical form $x = \xi t + \phi \xi$, then we know that

$$D_t = d_t + \xi d_x + \eta d_y + \zeta d_z + \dots,$$

and $\therefore$ also $F(D_t) = F(d_t + \xi d_x + \eta d_y + \zeta d_z + \dots)$ ...... (3).

20. The simplest case of quasi-constants is found by equating each of the functions $\phi_1, \phi_2, \dots$ to zero, in which case we have

$$\xi = \frac{x}{t}, \quad \eta = \frac{y}{t}, \quad \zeta = \frac{z}{t}, \quad \dots \text{\&c.} \dots\dots\dots\dots (4),$$

as the simplest set of quasi-constants that can be found.

21. If in the equation $D_t = d_t + Ld_x + Md_y + Nd_z + \dots$ some of the coefficients $L, M, N, \dots$ are absolute constants, and some of them quasi-constants, we shall have

$$F(D_t) = F(d_t + Ld_x + Md_y + Nd_z + \dots) \dots\dots\dots (5).$$

22. We may here make one general remark which will apply to great numbers of equations. If we have an equation involving differential symbols of operation of one kind only, as d_t, or D_t, or Δ_t, ... then any *absolute* constant in that equation may be replaced by a *quasi*-constant, without in any way affecting the validity of the integral. We shall illustrate this property of quasi-constants by an example or two; and afterwards it will be assumed, even when not mentioned, that the compass of every differential equation and its integral *may* be extended by means of this substitution of quasi-constants for absolute constants.

23. We may also mention, for form's sake, what is obvious enough, that in integrating an equation in which two or more differential symbols of different kinds occur, the quasi-con-

stants corresponding to one symbol will not all (though some of them *may*) be quasi-constants under the other differential symbols.

And, moreover, when the operating symbol in a differential equation is d_t, we may use functions of $(d_x, d_y, d_z ...)$ as quasi-constants.

Ex. 1. To integrate $td_tu + xd_xu + yd_yu = x^ay^bu^c$.

Dividing by t we see that the coefficients of d_xu and d_yu are quasi-constants (Art. 20), and that $\xi = \frac{x}{t}$, $\eta = \frac{y}{t}$;

$$\therefore \ D_tu = t^{-1}x^ay^bu^c;$$

$$\therefore \ u^{-c}D_tu = \xi^a\eta^b\, t^{a+b-1};$$

$$\therefore \ \frac{u^{1-c}}{1-c} = \xi^a\eta^b . \frac{t^{a+b}}{a+b} + F(\xi, \eta)$$

$$= \frac{x^ay^b}{a+b} + F\left(\frac{x}{t}, \frac{y}{t}\right).$$

Now this integral has been found on the supposition that a, b, c are absolute constants, but every step of the investigation will be equally valid if we suppose a, b, c to be functions of $\frac{x}{t}$, $\frac{y}{t}$, (Art. 22).

Ex. 2. To integrate $t^2d_t^2u + 2txd_td_xu + x^2d_x^2u = t^ax^b$.

Dividing by t^2 we find $D_t = d_t + \frac{x}{t} . d_x$, and consequently the coefficients are quasi-constants, and $\xi = \frac{x}{t}$.

$$\therefore \ D_t^2u = t^{a-2}x^b = \xi^b . t^{a+b-2};$$

$$\therefore\ u = \xi^b \cdot \frac{t^{a+b}}{(a+b)(a+b-1)} + tF(\xi) + f(\xi)$$

$$= \frac{t^a x^b}{(a+b)(a+b-1)} + tF\left(\frac{x}{t}\right) + f\left(\frac{x}{t}\right).$$

We are at liberty to write for a and b any functions of $\frac{x}{t}$, (Art. 22).

Ex. 3. To integrate

$$t^2 d_t^2 u + 2tx d_t d_x u + x^2 d_x^2 u + b(ht + kx)\left(d_t u + \frac{x}{t} d_x u\right) + cu = 0.$$

Here as in the last example $D_t = d_t + \frac{x}{t} d_x$, and $\xi = \frac{x}{t}$;

$$\therefore\ t^2 D_t^2 u + bt(h + k\xi) D_t u + cu = 0,$$

the integral of which is obtained as in ordinary integration for one independent variable;

$$\therefore\ u = t^m F(\xi) + t^n f(\xi)$$

$$= t^m F\left(\frac{x}{t}\right) + t^n f\left(\frac{x}{t}\right);$$

m and n being the two roots of the equation

$$m(m-1) + \frac{b}{t}(ht + kx)m + c = 0.$$

We are at liberty to write for b and c any functions of $\frac{x}{t}$.

Ex. 4. To integrate

$$d_t^2 u + 2a d_t d_x u + a^2 d_x^2 u + \frac{b}{t}(d_t u + a d_x u) + \frac{cu}{t^2} = 0.$$

In this example $D_t = d_t + ad_x$; $\therefore$ $D_t x = a$, and $x = at + \xi$.

$$\therefore\ D_t^2 u + \frac{b}{t} . D_t u + \frac{c}{t^2} . u = 0;$$

$$\begin{aligned}\therefore\ u &= t^m F(\xi) + t^n f(\xi)\\ &= t^m F(x - at) + t^n f(x - at);\end{aligned}$$

m and n being the roots of the equation

$$m(m-1) + bm + c = 0.$$

In this example we are at liberty to write for b and c any functions of $x - at$.

Ex. 5. To integrate $(ht + kx)^2 (d_t^2 u + 2ad_t d_x u + a^2 d_x^2 u)$

$$+ b(ht + kx)(d_t u + ad_x u) + cu = 0.$$

Here we assume $D_t = d_t + ad_x$, whence $\xi = x - at$;

also $$D_t (ht + kx) = h + ka;$$

$$\therefore\ (ht + kx)^2 D_t^2 u + b(ht + kx) D_t u + cu = 0.$$

This is an equation of one independent variable, because $D_t x = a$;

$$\begin{aligned}\therefore\ u &= (ht + kx)^m . F\xi + (ht + kx)^n . f\xi\\ &= (ht + kx)^m . F(x - at) + (ht + kx)^n . f(x - at),\end{aligned}$$

m, n being the values of m in the equation

$$(h + ak)^2 m(m-1) + b(h + ak)m + c = 0.$$

We might add here the same remark as before respecting the constants b, c.

24. The preceding examples will also establish the principle, that if any equation of one independent variable, $\phi(d_t)u = 0$, can be integrated, then the corresponding equa-

tion $\phi(D_t).u=0$ can be integrated, even when the quantities which were constant in $\phi(d_t).u=0$ are replaced in $\phi(D_t).u=0$ by quasi-constants, or in some cases by such quantities as $\psi(ht+kx)$. Thus because the integral of

$$d_t^2u+ad_tu+bu=0 \text{ is } u=A\epsilon^{mt}+B\epsilon^{nt},$$

m and n being the roots of the equation $m^2+am+b=0$; therefore the integral of

$$D_t^2u+aD_tu+bu=0$$

will be $$u=\epsilon^{mt}F(\xi,\eta,\ldots)+\epsilon^{nt}f(\xi,\eta,\ldots),$$

in which $\xi, \eta, \ldots$ are quasi-constants of any form we choose to assign to them.

The equation

$$d_t^2u+2\frac{x}{t}.d_td_xu+\frac{x^2}{t^2}.d_x^2u+a\left(d_tu+\frac{x}{t}d_xu\right)+bu=0,$$

may be taken in illustration of this principle.

Ex. 6. To integrate

$$td_tu+xd_xu+yd_yu+\ldots=t^n\phi\left(\frac{x}{t},\ \frac{y}{t},\ \ldots\right).$$

Dividing by t we have

$$\xi=\frac{x}{t},\ \eta=\frac{y}{t},\ldots \text{ and } D_tu=t^{n-1}\phi\left(\frac{x}{t},\ \frac{y}{t}\ldots\right);$$

$$\therefore\ D_tu=t^{n-1}\phi(\xi,\eta,\ldots);$$

$$\therefore\ u=\frac{t^n}{n}.\phi(\xi,\eta,\ldots)+F(\xi,\eta,\ldots)$$

$$=\frac{t^n}{n}.\phi\left(\frac{x}{t},\ \frac{y}{t},\ \ldots\right)+F\left(\frac{x}{t},\ \frac{y}{t},\ \ldots\right).$$

Ex. 7. To integrate

$$d_t u + \xi d_x u + \eta d_y u + \ldots = t^{n-1} \phi (\xi, \eta, \ldots).$$

As in the last example we have here

$$u = \frac{t^n}{n} . \phi (\xi, \eta, \ldots) + F (\xi, \eta, \ldots).$$

This differs from the preceding example in this, that the values of $\xi, \eta, \ldots$ are not given as there, but are to be determined from the equations

$$x = \xi t + f_1 (\xi, \eta, \ldots), \quad y = \eta t + f_2 (\xi, \eta, \ldots), \quad \&c.$$

Ex. 8. To integrate $d_t u - a d_x u + \frac{y}{t} d_y u = \frac{y^2}{tx}$.

This is a case where the coefficients are mixed constants. $D_t x = - a$, and therefore $x = - at + \xi$; but $\frac{y}{t} = \eta$ is a quasi-constant;

$$\therefore \; D_t u = \frac{y^2}{tx} = \frac{\eta^2 t}{\xi - at};$$

$$\therefore \; u = - \frac{\eta^2 t}{a} - \frac{\eta^2 \xi}{a^2} . \log (\xi - at) + F (\xi, \eta)$$

$$= - \frac{y^2}{at} - \frac{y^2}{a^2 t^2} (x + at) \log x + F \left(x + at, \frac{y}{t} \right).$$

CHAPTER V.

THE GENERAL LINEAR EQUATION OF THE SECOND ORDER. COEFFICIENTS CONSTANT.

25. EVERY linear differential equation of one independent variable, and of the second order with constant coefficients, is by custom taken to be integrable in finite terms; that is, the ultimate elementary forms to which that general equation can be reduced are accepted as integrable. Formulæ both algebraic and transcendental have been invented, —logarithmic, exponential and trigonometric, direct and inverse, and have been agreed upon for the purpose that an integral expressed in terms of such forms shall be accepted as a finite integral. Would it be too much to ask that whether there be only one or any number of independent variables, the elementary forms of the general equation of the second order with constant coefficients may be considered integrable forms, and that new transcendentals and appropriate symbols shall be invented for that purpose and take rank with log, sin, $\cos^{-1}$, ... ? To agree to this would only be consenting to put all integrations of the second order on the same level, whatever the number of variables.

We shall proceed to shew how on the principles of this Essay the general partial differential equation, which is the subject of this Chapter, may be reduced to its ultimate elementary forms. We shall take the case of two independent variables only, but we shall treat it in such a manner that the method will be equally applicable to any number of independent variables. We confine our operations to equations of two independent variables, because when there are more, the algebraic expressions become tediously long without illustrating any new principle.

26. The general equation of the second order embraces the two following forms:—

I. $d_t^2 u + (a+b) d_t d_x u + ab d_x^2 u + A d_t u + B d_x u + Cu = W.$

II. $d_t d_x u + A d_t u + B d_x u + Cu = W.$

We begin with the former.

Case I. When a and b are not equal.

Since $d_t^2 + (a+b)\, d_t d_x + ab d_x^2 = (d_t + a d_x)\,(d_t + b d_x),$

we assume $D_t = d_t + a d_x,\quad \Delta_t = d_t + b d_x;$

$$\therefore\ D_t x = a,\quad x = at + \xi;\ \text{and}\ \Delta_t x = b,\quad x = bt + \xi' :$$

and the proposed equation becomes

$$D_t \Delta_t u + \frac{B - Ab}{a - b} \,.\, D_t u + \frac{B - Aa}{b - a} \,.\, \Delta_t u + Cu = W \ \ldots\ldots (1);$$

$$\therefore \left(D_t + \frac{B-Aa}{b-a}\right)\left(\Delta_t + \frac{B-Ab}{a-b}\right) u + \left\{C + \frac{(B-Aa)(B-Ab)}{(a-b)^2}\right\} u = W (2).$$

Now for u write $v \epsilon^{m\xi' + n\xi}$, remembering that $D_t \xi = 0$, $\Delta_t \xi' = 0$, $D_t \xi' = a - b$, and $\Delta_t \xi = b - a$.

$$\therefore \left(D_t + \frac{B - Aa}{b - a}\right) u = \epsilon^{m\xi' + n\xi} \left\{D_t + m(a - b) + \frac{B - Aa}{b - a}\right\} . v,$$

and $$\left(\Delta_t + \frac{B - Ab}{a - b}\right) u = \epsilon^{m\xi' + n\xi} \left\{\Delta_t + n(b - a) + \frac{B - Ab}{a - b}\right\} . v.$$

As m, n are arbitrary constants we are at liberty to assume

$$(a - b)^2 m = B - Aa, \text{ and } (a - b)^2 n = B - Ab;$$

$$\therefore D_t \Delta_t v + \left\{C + \frac{(B - Aa)(B - Ab)}{(a - b)^2}\right\} v = W\epsilon^{-m\xi' - n\xi}.$$

Consequently when $W = 0$, the form I. can be reduced to the elementary form,

$$D_t \Delta_t v = cv.$$

Case II. When a and b are equal.

In this case the form I. becomes

$$(d_t + ad_x)^2 u + A\left(d_t + \frac{B}{A} d_x\right) u + Cu = W.$$

Hence $D_t = d_t + ad_x$, $x = at + \xi$ as before; and $\Delta_t = d_t + \frac{B}{A} d_x$, whence $\Delta_t x = \frac{B}{A}$, and $x = \frac{B}{A} t + \xi'$;

$$\therefore D_t^2 u + A\Delta_t u + Cu = W.$$

Now for u write $v\epsilon^{m\xi}$, remembering that $\Delta_t \xi = \frac{B}{A} - a$;

$$\therefore D_t^2 v + A\Delta_t v + \{(B - Aa) m + C\} v = W\epsilon^{-m\xi}.$$

We are at liberty to assume

$$(Aa - B) m = C;$$

$$\therefore D_t^2 v + A\Delta_t v = W\epsilon^{-m\xi}.$$

Consequently when $W = 0$, the form I. when $a = b$ can be reduced to the elementary form,

$$D_t^2 v = c\Delta_t v.$$

There is however an exception to this result, viz. when $B = Aa$, for then the original equation becomes of the following form:

$$D_t^2 u + AD_t u + Cu = W,$$

which is always integrable in finite terms (Art. 24).

Case III. When $d_t d_x u + Ad_t u + Bd_x u + Cu = W$;

$$\therefore\ (d_t + B)\ (d_x + A)\, u + (C - AB)\, u = W.$$

For u write $v\epsilon^{mt+nx}$, and assume $m + B = 0$, $n + A = 0$, then

$$u = v^{-Ax-Bt},$$

$$\text{and } d_t d_x v + (C - AB)\, v = W\epsilon^{Ax+Bt};$$

consequently when $W = 0$, the form II. can be reduced to the elementary form,

$$d_t d_x v = cv,$$

which is the same as in the first case.

Thus it appears that there are two elementary forms, and no more, viz.

$$D_t \Delta_t u = cu, \text{ and } D_t^2 u = c\Delta_t u.$$

The question is—are we to accept these as integrable forms? Their symbolical integrals can be easily found, but they cannot be generally integrated in finite terms.

Ex. 1. To find the symbolical integral of $D_t \Delta_t u = cu$.

Since D_t and Δ_t are independent symbols, each of them may be treated as a constant in reference to the other;

$$\therefore\ D_t u = \frac{c}{\Delta_t} \,.\, u = c\Sigma_t \,.\, u\,;$$

$$\therefore\ u = \epsilon^{ct\Sigma_t}\, F(\xi).$$

If it should be thought that there is any ambiguity in this expression, we may exhibit it in another form, free from all ambiguity, by considering that

$$\Delta_t F(\xi) = F'(\xi) \,.\, \Delta_t \xi = d_\xi F(\xi) \,.\, (b-a)\,;$$

$$\therefore\ (b-a)\,\Sigma_t = s_\xi\,;$$

$$\therefore\ u = \epsilon^{\frac{cts_\xi}{b-a}}\, F(\xi).$$

In a similar way we should find

$$u = \epsilon^{\frac{cts_{\xi'}}{a-b}} f(\xi').$$

Consequently the complete symbolical integral is

$$u = \epsilon^{\frac{ct}{b-a}\cdot s_\xi}\, F(\xi) + \epsilon^{\frac{ct}{a-b}\cdot s_{\xi'}}\, f(\xi').$$

Remark. If in these symbolic exponentials we had d_ξ, $d_{\xi'}$, instead of s_ξ, $s_{\xi'}$, the value of u would be expressible in finite terms. Why not extend this character of finiteness to the above form by the invention of an appropriate transcendental?

Ex. 2. To find the symbolical integral of $D_t^2 u = c\Delta_t u$.

Since Δ_t is constant in reference to D_t, we have for the partial integral of this equation,

$$u = \left(\epsilon^{t\sqrt{c\Delta_t}} + \epsilon^{-t\sqrt{c\Delta_t}}\right) F(\xi) + \left(\epsilon^{t\sqrt{c\Delta_t}} - \epsilon^{-t\sqrt{c\Delta_t}}\right) \,.\, \frac{f(\xi)}{\sqrt{c\Delta_t}}\,.$$

The connexion between these two terms is this:—if the second be represented by u, the first will be represented by $D_t u$ with a change of the arbitrary function. As the proposed equation is linear D_t^2u, D_t^3u, ... each with a change of the arbitrary function will be an integral; but as D_t^2u will coincide in form with u, and D_t^3u with D_tu, there are only two forms which are different, and the sum of these two given above constitute the whole integral of u as far as the differential equation depends on D_t.

But since the differential equation depends also on Δ_t the integral must contain an arbitrary function of ξ'. To obtain this we put the equation in the form

$$\Delta_t u = \frac{D_t^2}{c} \,.\, u\,;$$

$$\therefore\ u = \epsilon^{\frac{t}{c}\,.\,D_t^2}\,\phi\,(\xi').$$

The complete integral will be obtained by adding this to the above.

27. But now we come upon a difficulty. If we add this to the preceding portion of the integral, we appear to have an integral with *three* instead of *two* arbitrary constants. Will it be sufficient to take the sum of this with *one* of the former two? Do two of the three terms of the whole include the other term? The probability of this may perhaps be due to the circumstance that each term of the integral represents in reality an infinite series, and that in the whole integral each arbitrary function enters with all its derived forms.

Since in the symbolical integral each arbitrary function enters with all its derived forms, the finiteness of the integral can be secured only by assigning to each arbitrary function a periodic form. It is easy to shew that the integral cannot be finite on any other supposition. For if possible let the integral of $D_t^2u = c\Delta_tu$ be finite, and let it involve the arbitrary functions $F(\xi)$ and $f(\xi')$. Then $D_t^2 u$ will involve $f(\xi')$, $f'(\xi')$ and $f''(\xi')$ in addition to $F(\xi)$; it will not contain $F'(\xi)$, $F''(\xi)$, because ξ is constant in reference to D_t.

The other side of the equation $c\Delta_t u$ will involve $F(\xi)$, $F'(\xi)$ in addition to $f(\xi')$. Hence when we substitute for u its value in terms of $F(\xi)$ and $f(\xi')$, we shall have on the left-hand side of the equation $f'(\xi')$, $f''(\xi')$, which are not to be found at all on the right-hand side. On the right hand also there will be $F'(\xi)$, which does not exist on the left hand; and as these functions are arbitrary, and not periodic, the two sides of the equation never can be equivalent to each other. If then a finite integral is required, periodic forms must be found for the arbitrary functions.

28. The following integrals are easily found on this principle. They are given for the sake of reference and illustration.

(1) Given $\xi = x - at$, $\xi' = x - bt$, to find a finite integral of the equation $D_t \Delta_t u = cu$.

$$u = C_1 \epsilon^{\frac{\sqrt{c}}{a-b} \cdot (m\xi + n\xi')} + C_2 \epsilon^{\frac{\sqrt{c}}{b-a} \cdot (m\xi + n\xi')}.$$

C_1, C_2, m and n being arbitrary constants, the last two being subject to the condition $mn + 1 = 0$.

(2) Given $\xi = x - at$, and $\xi' = x - bt$, to find a finite integral of the equation $D_t^2 u = c\Delta_t u$.

$$u = (C_1 \epsilon^{n\sqrt{c} . \xi'} + C_2 \epsilon^{-n\sqrt{c} . \xi'}) . \epsilon^{n^2 (a-b) \xi} + (C_3 \epsilon^{n\sqrt{c} . \xi'} + C_4 \epsilon^{-n\sqrt{c} . \xi'}) . \epsilon^{n^2 (b-a) \xi}.$$

C_1, C_2, C_3, C_4, n are all arbitrary constants.

(3) Given $d_t^2 u = a^2 d_x^2 u + b^2 d_y^2 u$, to find an integral in finite terms.

This equation gives

$$(d_t + ad_x)(d_t - ad_x) u = b^2 d_y^2 u.$$

Hence $D_t = d_t + a d_x$, $\Delta_t = d_t - a d_x$, $\xi = x - at$, $\xi' = x + at$;

$$\therefore\ D_t \Delta_t u = d_y^2 u.$$

The symbolical integral may be at once obtained from this by integrating separately with S_t, Σ_t, s_y; and it will involve four arbitrary functions,

$$F(\xi, y),\ f(\xi', y),\ \phi_1(\xi, \xi'),\ \phi_2(\xi, \xi').$$

But the following integral in finite terms, though involving only two arbitrary functions, may be useful in certain cases.

$$u = F\left(m\xi + \frac{\xi'}{m} + \frac{2a}{b} \cdot y\right) + f\left(m\xi + \frac{\xi'}{m} - \frac{2a}{b} \cdot y\right),$$

m being an arbitrary constant.

The same may be given in the following form:

$$u = F\left(m\xi - \frac{\xi'}{m} + \frac{2a\sqrt{-1}}{b} \cdot y\right) + f\left(\frac{\xi'}{m} - m\xi + \frac{2a\sqrt{-1}}{b} \cdot y\right),$$

and these two forms may be combined by addition with others derived from these by writing $\frac{1}{m}$ for m.

(4) The equation $d_t^2 u + d_x^2 u = cu$ by a change of the independent variables may be reduced to the form $d_t d_x u = cu$.

The symbolical integral of the latter form is

$$u = \epsilon^{cts_x} F(t) + \epsilon^{cxs_t} f(x).$$

And the following is an integral in finite terms,

$$u = C_1 \epsilon^{\left(mt + \frac{x}{m}\right)\sqrt{c}} + C_2 \epsilon^{-\left(mt + \frac{x}{m}\right)\sqrt{c}},$$

C_1, C_2, m being all arbitrary constants.

If c be negative, the finite integral of $d_t d_x u + cu = 0$ is

$$u = C_1 \epsilon^{\left(mt - \frac{x}{m}\right)\sqrt{c}} + C_2 \epsilon^{\left(\frac{x}{m} - mt\right)\sqrt{c}}.$$

In this and the preceding case we are at liberty to add other terms to the finite integrals, derived from those given by changing m into $\frac{1}{m}$.

The equation $d_t^2 u - a^2 d_x^2 u = cu$, being treated in the usual manner, gives $D_t = d_t + a d_x$, $\Delta_t = d_t - a d_x$, $\xi = x - at$, $\xi' = x + at$;

$$\therefore \; D_t \Delta_t u = cu.$$

This result shews that it is vain to look for a finite general integral of any equation of the form $d_t^2 u - a^2 d_x^2 u = cu$.

Ex. 3. To integrate

$$d_t^2 u + (a + b)\, d_t d_x u + ab d_x^2 u + A d_t u + B d_x u + Cu = W,$$

when the coefficients satisfy the equation

$$C(a - b)^2 + (B - Aa)(B - Ab) = 0.$$

On reference to equation (2) Art. 26, Case I., it will be seen that when the above condition is satisfied by the constants, the proposed equation may be reduced to

$$\left(D_t + \frac{B - Aa}{b - a}\right)\left(\Delta_t + \frac{B - Ab}{a - b}\right) u = W.$$

For brevity write

$$(D_t + \alpha)\,(\Delta_t + \beta)\, u = W;$$

$$\therefore \; \epsilon^{-\alpha t} D_t \epsilon^{\alpha t} \,.\, \epsilon^{-\beta t} \Delta_t \epsilon^{\beta t} u = W;$$

$$\therefore \; u = \epsilon^{-\alpha t} S_t \epsilon^{\alpha t - \beta t} \Sigma_t \epsilon^{\beta t} W + \epsilon^{-\alpha t} S_t \epsilon^{\alpha t} 0 + \epsilon^{-\beta t} \Sigma_t \epsilon^{\beta t} 0$$

$$= \epsilon^{-\alpha t} S_t \epsilon^{(\alpha - \beta)t} \Sigma_t \epsilon^{\beta t} W + \epsilon^{-\alpha t} F(\xi) + \epsilon^{-\beta t} f(\xi'),$$

in which $\xi = x - at$ and $\xi' = x - bt$.

Ex. 4. To integrate

$$d_t^2u + 2ad_td_xu + a^2d_x^2u + A\ (d_tu + ad_xu) + Cu = W.$$

This is equivalent to

$$D_t^2u + AD_tu + Cu = W;$$

$$\therefore\ (D_t + \alpha)\ (D_t + \beta)\ u = W;$$

if α, β be the roots of $a^2 - Aa + C = 0$.

$$\therefore\ u = \epsilon^{-\alpha t}S_t\epsilon^{(\alpha-\beta)t}S_t\epsilon^{\beta t}W + \epsilon^{-\alpha t}F(\xi) + \epsilon^{-\beta t}f(\xi).$$

The double integration in the first term of this integral may be avoided.

In this example A, C may be quasi-constants (Art. 22).

29. The resolution of multiple integration into single integration may be thus effected.

Since $\qquad (d_t + \alpha)\ W = \epsilon^{-\alpha t}d_t\epsilon^{\alpha t}W,$

and $\qquad (d_t + \beta)\ W = \epsilon^{-\beta t}d_t\epsilon^{\beta t}W;$

$$\therefore\ (\beta - \alpha)\ W = \epsilon^{-\beta t}d_t\epsilon^{\beta t}W - \epsilon^{-\alpha t}d_t\epsilon^{\alpha t}W.$$

Integrate both sides with $\epsilon^{-\alpha t}s_t\epsilon^{\alpha t}\,.\,\epsilon^{-\beta t}s_t\epsilon^{\beta t}$;

$$\therefore\ (\beta - \alpha)\ \epsilon^{-\alpha t}s_t\epsilon^{\alpha t-\beta t}s_t\epsilon^{\beta t}W = \epsilon^{-\alpha t}s_t\epsilon^{\alpha t}W - \epsilon^{-\beta t}s_t\epsilon^{\beta t}W.$$

This is the formula for the reduction of a double integral to two single ones.

If we integrate both sides with $\epsilon^{-\gamma t}s_t\epsilon^{\gamma t}$ the left hand will become a triple integral, and the right hand two double integrals. Each of the latter may be resolved into single integrals, and the final result will be found to be

$$\epsilon^{-\alpha t}s_t\epsilon^{(\alpha-\beta)t}s_t\epsilon^{(\beta-\gamma)t}s_t\epsilon^{\gamma t}W = \frac{\epsilon^{-\alpha t}s_t\epsilon^{\alpha t}W}{(\gamma-\alpha)(\beta-\alpha)} + \frac{\epsilon^{-\beta t}s_t\epsilon^{\beta t}W}{(\gamma-\beta)(\alpha-\beta)} + \frac{\epsilon^{-\gamma t}s_t\epsilon^{\gamma t}W}{(\beta-\gamma)(\alpha-\gamma)}.$$

The law is obvious for any number of integrations.

If any of the quantities α, β, γ, ... be equal, the method will hold good, but the formula just found will fail. The method will furnish the right formula in every case.

Thus if $\alpha=\beta$, $\epsilon^{-\alpha t}s_t\epsilon^{\alpha t} \,.\, \epsilon^{-\alpha t}s_t\epsilon^{\alpha t} W=\epsilon^{-\alpha t}s_t^2\epsilon^{\alpha t} W$, which does not admit of further reduction.

By putting $W=0$ we obtain the following formula, which agrees with Art. 16,

$$\epsilon^{-\alpha t}s_t\epsilon^{\alpha t} \,.\, \epsilon^{-\beta t}s_t\epsilon^{\beta t} \,.\, \epsilon^{-\gamma t}s_t\epsilon^{\gamma}\, 0=\epsilon^{-\alpha t}s_t 0+\epsilon^{-\beta t}s_t 0+\epsilon^{-\gamma t}s_t 0.$$

Ex. 5. To integrate $d_t d_x u+A d_t u+B d_x u+ABu=W$.

This is equivalent to $(d_t+B)\,(d_x+A)\,u=W$;

$$\therefore\ \epsilon^{-Bt}d_t\epsilon^{Bt} \,.\, \epsilon^{-Ax}d_x\epsilon^{Ax}u=W;$$

$$\therefore\ u=\epsilon^{-Bt}s_t\epsilon^{Bt-Ax}s_x\epsilon^{Ax}W+\epsilon^{-Bt}s_t\epsilon^{Bt}0+\epsilon^{-Ax}s_x\epsilon^{Ax}0$$

$$=\epsilon^{-Bt}s_t\epsilon^{Bt-Ax}s_x\epsilon^{Ax}W+\epsilon^{-Bt}F(x)+\epsilon^{-Ax}f(t).$$

Ex. 6. To integrate

$$d_t^2u+2ad_td_xu+a^2d_x^2u=\phi''\,(ht+kx,\ \xi)\,;$$

where ϕ'' is the second differential coefficient of ϕ with regard to $ht+kx$, ξ being considered constant.

In this case

$$D_t=d_t+ad_x,\ \ \xi=x-at,\ \text{ and }\ D_t\,(ht+kx)=h+ka\,;$$

$$\therefore\ D_t^2u=\phi''\,(ht+kx,\ \xi)\,;$$

$$\therefore\ u=\frac{\phi\,(ht+kx,\ \xi)}{(h+ka)^2}+tF(\xi)+f(\xi).$$

Ex. 7. To integrate

$$d_t^2u+\frac{2x}{t}\,.\,d_td_xu+\frac{x^2}{t^2}\,.\,d_x^2u=\phi''\,(ht+kx,\ \xi).$$

In this case

$$D_t = d_t + \frac{x}{t} \cdot d_x, \quad \xi = \frac{x}{t}, \text{ and } D_t(ht + kx) = h + k \cdot \frac{x}{t};$$

$$\therefore\ D_t^2 u = \phi''(ht + kx,\ \xi);$$

$$\therefore\ u = \frac{t^2}{(ht + kx)^2} \cdot \phi(ht + kx,\ \xi) + tF\left(\frac{x}{t}\right) + f\left(\frac{x}{t}\right).$$

Ex. 8. To integrate

$$d_t^2 u + 2\xi d_t d_x u + \xi^2 d_x^2 u = \phi''(ht + kx,\ \xi).$$

In this case $x = \xi t + \phi_1(\xi)$ ....(1).

$$u = \frac{\phi(ht + kx)}{(h + k\xi)^2} + tF(\xi) + f(\xi).$$

ξ in this Example is any quantity which satisfies equation (1); and the function ϕ_1 is arbitrary.

Several of the examples contained in this Chapter will be more fully treated of in the last Chapter.

CHAPTER VI.

EQUATIONS OF THE SECOND ORDER WITH VARIABLE COEFFICIENTS.

30. As our object is to exhibit the new method of integration, we shall, in order to avoid complexity, produce formulæ for only two independent variables, but the methods used will always be equally applicable to any number of variables.

$$\text{Since } D_t u = d_t u + D_t x \,.\, d_x u\,;$$

$$\therefore\ \Delta_t D_t u = \Delta_t\,(d_t u) + \Delta_t\,(D_t x)\,.\,d_x u + D_t x\,.\,\Delta_t\,(d_x u).$$

By writing $d_t u$, $d_x u$ successively for u in the former equation, and substituting the results in the latter, we have

$$\Delta_t D_t u = d_t^2 u + (D_t x + \Delta_t x)\, d_t d_x u + D_t x\,.\,\Delta_t x\,.\,d_x^2 u \\ + \Delta_t D_t x\,.\,d_x u \;\ldots\ldots\ldots\ldots\; (1).$$

This is a very important formula, and will enable us in solving a certain class of equations with variable coefficients to eliminate at one step all terms of the second order, by substituting $\Delta_t D_t u - \Delta_t D_t x\,.\,d_x u$ for them.

31. It is important to notice that if the operative symbols $D_t \Delta_t$ be commutative on x, they will be so on u, i.e. on any function of t, x. The commutativeness of the symbols of

operation is of such great utility in the work of integration, that we shall prove this proposition for any number of variables.

Let u be represented by $\phi(x)$ as far as it depends on the independent variable x. Then as far as x is concerned,

$D_t u = \phi'(x) \,.\, D_t x$, and $\Delta_t D_t u = \phi'(x) \,.\, \Delta_t D_t x + \phi''(x) \,.\, \Delta_t x \,.\, D_t x.$

But by performing the operations in a contrary order we should find

$$D_t \Delta_t u = \phi'(x) \,.\, D_t \Delta_t x + \phi''(x) \,.\, D_t x \,.\, \Delta_t x.$$

Consequently if $\Delta_t D_t x = D_t \Delta_t x$, i.e. if D_t, Δ_t be commutative on x, then as far as u depends on x, they are commutative on u, any function of the variable. The same may be proved for $y, z, \ldots$ the remaining variables. Hence the symbols of operation D_t, Δ_t are commutative on u in reference to all the variables on which they are individually commutative.

As a general rule, also, the symbols will be commutative on any variable, as x, if $D_t x$ and $\Delta_t x$ be both constant.

32. When it is desirable to commutate the symbols D_t, Δ_t, and they are not commutative *per se*, they may be commutated by the formula,

$$D_t \Delta_t u = \Delta_t D_t u + (D_t \Delta_t x - \Delta_t D_t x) \,.\, d_x u.$$

For since

$$\Delta_t D_t u = d_t^2 u + (D_t x + \Delta_t x) \,.\, d_t d_x u + D_t x \,.\, \Delta_t x \,.\, d_x^2 u + \Delta_t D_t x . d_x u\,;$$

$$\therefore\ D_t \Delta_t u = d_t^2 u + (\Delta_t x + D_t x) . d_t d_x u + \Delta_t x . D_t x . d_x^2 u + D_t \Delta_t x . d_x u$$
$$= \Delta_t D_t u + (D_t \Delta_t x - \Delta_t D_t x) \,.\, d_x u.$$

33. The following formulæ are easily proved, P, Q being any functions of the independent variables.

$$\epsilon^{-P} \Delta_t \epsilon^P D_t u = d_t^2 u + (D_t x + \Delta_t x)\, d_t d_x u + D_t x . \Delta_t x . d_x^2 u$$
$$+ \Delta_t P . d_t u + (\Delta_t P . D_t x + \Delta_t D_t x) . d_x u \ldots\ldots\ldots\ldots (2).$$

$$\epsilon^{-P}\Delta_t\epsilon^{P-Q}D_t\epsilon^{Q}u = d_t^2u + (D_tx + \Delta_tx).d_td_xu + D_tx.\Delta_tx.d_x^2u$$
$$+ (\Delta_tP + D_tQ).d_tu + (\Delta_tP.D_tx + D_tQ.\Delta_tx + \Delta_tD_tx)\,d_xu$$
$$+ (\Delta_tP.D_tQ + \Delta_tD_tQ)\,u \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

34. It will be seen that the formulæ (1), (2), (3) increase step by step in comprehensiveness. Formula (1) can, by giving proper values to D_tx, Δ_tx (which are arbitrary), be made to agree with all the terms of the second order in any proposed equation. Formula (2) can be made to agree with one term more by giving a proper value to P (which is arbitrary). Formula (3) can be made to agree with one term more by assigning to Q a proper value. If therefore we have an equation of the general form

$$d_t^2u + Hd_td_xx + Kd_x^2u + Ld_tu + Md_xu + Ru = W,$$

it can always be reduced to the form

$$\epsilon^{-P}\Delta_t\epsilon^{P-Q}D_t\epsilon^{Q}u + R'u = W,$$

and if R' be equal to zero, it can be integrated.

35. The formula (1) shews that we can substitute

$$\Delta_tD_tu - (\Delta_tD_tx)\,.\,d_xu$$

for the whole of the terms of the second order; and since

$$D_tu = d_tu + D_tx\,.\,d_xu, \text{ and } \Delta_tu = d_tu + \Delta_tx\,.\,d_xu,$$

these two equations will enable us to eliminate d_tu and d_xu from the remaining part of any proposed equation.

Ex. 1. To integrate

$$d_t^2u + 2ad_td_xu + (a^2 - b^2t^2)\,d_x^2u - bd_xu = W.$$

Comparing this equation with formula (1) we find

$$D_tx + \Delta_tx = 2a, \text{ and } D_tx\,.\,\Delta_tx = a^2 - b^2t^2;$$

$\therefore\ D_t x = a - bt$, and $\Delta_t x = a + bt$; and $\therefore\ \Delta_t D_t x = -b$;

$$\therefore\ x = at - \frac{1}{2}bt^2 + \xi, \text{ and } x = at + \frac{1}{2}bt^2 + \xi';$$

$$\therefore\ \Delta_t D_t u = W;$$

$$\therefore\ D_t u = \Sigma_t W + f(\xi');$$

$$\therefore\ u = S_t \Sigma_t W + S_t f(\xi') + F(\xi)$$

$$= S_t \Sigma_t W + S_t f(\xi - bt^2) + F\left(x - at + \frac{1}{2}bt^2\right).$$

REMARK. In this example $\Delta_t D_t x = -b$, and $D_t \Delta_t x = b$; consequently $\Delta_t D_t x$ is not equal to $D_t \Delta_t x$, and D_t, Δ_t are not commutative. As an instance of the important influence of commutativeness of the symbols D_t, Δ_t, we may mention that had D_t, Δ_t been commutative the absolute part of the above integral would have been

$$S_t O + \Sigma_t O = F(\xi) + f(\xi') = F\left(x - at + \frac{1}{2}bt^2\right)$$
$$+ f\left(x - at - \frac{1}{2}bt^2\right).$$

Ex. 2. To integrate $\dfrac{d_t^2 u}{t^2} - \dfrac{d_x^2 u}{x^2} = \dfrac{d_t u}{t^3} - \dfrac{d_x u}{x^3} + W$.

Multiply this equation by t^2,

$$\therefore\ d_t^2 u - \frac{t^2}{x^2}\,.\,d_x^2 u - \frac{1}{t}\,.\,d_t u + \frac{t^2}{x^3}\,.\,d_x u = Wt^2.$$

Now by formula (1)

$$\Delta_t D_t u - \Delta_t D_t x\,.\,d_x u = d_t^2 u + (D_t x + \Delta_t x)\, d_t d_x u + D_t x\,.\,\Delta_t x\,.\,d_x^2 u;$$

and this will agree with the terms of the second order of the proposed equation if we assume $D_t x = -\Delta_t x = \dfrac{t}{x}$;

$$\therefore\ x^2 = t^2 + \xi,\ x^2 = -t^2 + \xi',\ \text{and}\ \Delta_t D_t x = \frac{1}{x} + \frac{t^2}{x^3};$$

$$\therefore\ \Delta_t D_t u - \left(\frac{1}{x} + \frac{t^2}{x^3}\right) . d_x u - \frac{1}{t} . d_t u + \frac{t^2}{x^3} . d_x u = Wt^2;$$

$$\therefore\ \Delta_t D_t u - \frac{1}{t}\left(d_t u + \frac{t}{x} . d_x u\right) = Wt^2;$$

$$\therefore\ \Delta_t D_t u - \frac{1}{t} D_t u = Wt^2.$$

Divide by t and integrate with Σ_t;

$$\therefore\ D_t u = t\Sigma_t(Wt) + tf(\xi');$$

$$\therefore\ u = S_t t\Sigma_t(Wt) + S_t tf(\xi') + F(\xi).$$

But $S_t tf(\xi') = S_t tf(\xi + 2t^2) = \phi(\xi + 2t^2) = \phi(x^2 + t^2)$;

$$\therefore\ u = S_t t\Sigma_t Wt + F(x^2 - t^2) + \phi(x^2 + t^2).$$

Ex. 3. To integrate $(1 - t^2)^2 d_t^2 u + 2(1 - t^2)(1 - tx) d_t d_x u + (1 - tx)^2 d_x^2 u - 2t(1 - t^2) d_t u - (t + x - 2t^2 x) d_x u + c^2 u = 0.$

Divide by $(1 - t^2)^2$;

$$\therefore\ d_t^2 u + \frac{2(1 - tx)}{1 - t^2} . d_t d_x u + \left(\frac{1 - tx}{1 - t^2}\right)^2 d_x^2 u - \frac{2t}{1 - t^2} . d_t u - \frac{t + x - 2t^2 x}{(1 - t^2)^2} d_x u + \frac{c^2 u}{(1 - t^2)^2} = 0.$$

Comparing this with formula (1) we see immediately that the symbols D_t and Δ_t must be identical, and that

$$D_t x = \frac{1 - tx}{1 - t^2},\ \text{and}\ \therefore\ D_t^2 x = \frac{t - x}{(1 - t^2)^2};$$

$$\therefore\ D_t^2 u - \frac{t - x}{(1 - t^2)^2} . d_x u - \frac{2t}{1 - t^2} . d_t u - \frac{t + x - 2t^2 x}{(1 - t^2)^2} . d_x u + \frac{c^2 u}{(1 - t^2)^2} = 0;$$

$$\therefore\ D_t^2u - \frac{2t}{1-t^2}\cdot\left(d_tu + \frac{1-tx}{1-t^2}\cdot d_xu\right) + \frac{c^2u}{(1-t^2)^2} = 0\,;$$

$$\therefore\ D_t^2u - \frac{2t}{1-t^2}\cdot D_tu + \frac{c^2u}{(1-t^2)^2} = 0\,;$$

$$\therefore\ D_t\{(1-t^2)\,D_tu\} + \frac{c^2u}{1-t^2} = 0\,;$$

$$\therefore\ \{(1-t^2)\,D_t\}^2u + c^2u = 0.$$

Now change the independent to τ, such that $D_\tau t = 1 - t^2$.

$$\therefore\ D_\tau^2u + c^2u = 0\,;$$

$$\therefore\ u = F(\xi)\cos c\tau + f(\xi)\sin c\tau.$$

ξ is known from $(1-t^2)\,D_tx = 1 - tx$, and τ from $D_\tau t = 1 - t^2$.

$$\therefore\ \xi = \frac{x-t}{\sqrt{1-t^2}}, \text{ and } \tau = \log\sqrt{\frac{1+t}{1-t}}.$$

CHAPTER VII.

INTEGRATION BY FACTORIAL SYMBOLS OF OPERATION.

36. If T be any function of t, and $f(d_t)$ or f any function of the symbol of operation d_t, then the symbol $f(d_t)$ applied to TU is equivalent to the symbol $Tf + T'f' + \frac{1}{1.2} T''f'' + \frac{1}{1.2.3} . T'''f''' + \dots$, applied to U.

Since

$$d_t^n . TU = Td_t^n U + \frac{n}{1} d_t T . d_t^{n-1} U + \frac{n(n-1)}{1.2} d_t^2 T . d_t^{n-2} U + \dots$$

$$= (Td_t^n + d_t T . n d_t^{n-1} + d_t^2 T . n \frac{n-1}{2} d_t^{n-2} + \dots) \, U,$$

$$= (T . \kappa^n + T' . d_\kappa . \kappa^n + \frac{1}{1.2} T'' . d_\kappa^2 . \kappa^n + \dots) . U,$$

where the Greek letter κ is used to represent d_t as a symbol of *quantity*.

This being true for all integer values of n, it follows for all interpretable forms of $f(\kappa)$, that

$$f(d_t) \,.\, TU = \{Tf(\kappa) + T'd_\kappa f(\kappa) + \frac{1}{1\,.\,2} \,.\, T''d_\kappa^2 f(\kappa) + \dots\} \,.\, U,$$

$$= (Tf + T'f' + \frac{1}{1\,.\,2} \,.\, T''f'' + \frac{1}{1\,.\,2\,.\,3} T'''f''' + \dots) \,.\, U.$$

37. This theorem may be symbolically represented by the exponential expression,

$$f(d_t) \,.\, TU = \epsilon^{d_t d_\kappa} Tf(\kappa) \,.\, U,$$

in the use of which we are to notice that d_t in the exponential acts only on T, and d_κ only on $f(\kappa)$: and before the series of operations is to be applied to U the value of κ is to be restored, i.e. d_t is to be written for κ.

38. If P be a function of several independent variables $t, x, y, \dots$ and $f(d_t, d_x, \dots)$ any function of the operative symbols, then the symbol $f(d_t, d_x, \dots)$ applied to PU is equivalent to the symbol $\epsilon^{d_t d_\kappa + d_x d_\lambda + \dots} Pf(\kappa, \lambda, \dots)$ applied to U, the Greek letters $\kappa, \lambda, \dots$ representing $d_t, d_x \dots$ when these are to be treated as symbols of quantity.

For $$d_t^n \,.\, PU = \epsilon^{d_t d_\kappa} P\kappa^n \,.\, U,$$

$$\therefore\ d_x^m d_t^n \,.\, PU = d_x^m \epsilon^{d_t d_\kappa} P\kappa^n \,.\, U,$$

$$= \epsilon^{d_t d_\kappa} d_x^m P\kappa^n \,.\, U,$$

$$= \epsilon^{d_t d_\kappa} \epsilon^{d_x d_\lambda} P\kappa^n \lambda^m \,.\, U,$$

$$= \epsilon^{d_t d_\kappa + d_x d_\lambda} P\kappa^n \lambda^m \,.\, U.$$

Hence for all interpretable forms of $f(d_t, d_x)$ we have

$$f(d_t, d_x) \,.\, PU = \epsilon^{d_t d_\kappa + d_x d_\lambda} Pf(\kappa, \lambda) \,.\, U.$$

The same process may be continued to any number of independent variables; and consequently the theorem is true for any number.

39. If P be any function of the independent variables $t, x, y, z, \ldots$ and if $f(D_t, \Delta_t, \delta_t \ldots)$ be any interpretable function of the operative symbols when these are commutative among themselves, then the symbol $f(D_t, \Delta_t, \delta_t, \ldots)$ applied to PU is equivalent to the symbol $\epsilon^{D_t d_\kappa + \Delta_t d_\lambda + \delta_t d_\mu + \cdots} Pf(\kappa, \lambda, \mu, \ldots)$ applied to U; $\kappa, \lambda, \mu, \ldots$ representing $D_t, \Delta_t, \delta_t \ldots$ when these are to be treated as symbols of quantity.

The proof of this theorem is similar in principle to that given in the preceding article, and it is unnecessary to repeat it.

The cases of the theorems of this and the preceding articles of this chapter which most frequently occur are the following

$$(1) \ldots f(d_t) \,.\, tU = (tf + f')\, U.$$

$$(2) \ldots f(d_t \,.\, d_x) \,.\, (ht + kx)\, U = \{(ht + kx) f + (kd_t + hd_x) f'\} . U.$$

$$(3) \ldots\ldots f(D_t \Delta_t) \,.\, tU = \{tf + (D_t + \Delta_t) f'\} \,.\, U.$$

40. Examples of the use of these theorems will now be given.

Ex. 1. To integrate

$$d_t^2 u + (a + b)\, d_t d_x u + abd_x^2 u + Ad_t u + Bd_x u + Cu = \frac{C'}{t^2}\, u.$$

By proceeding as in Art. 26, Case I, we reduce this equation to the form

$$D_t \Delta_t v + cv = \frac{C'}{t^2} \,.\, v, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which equation $v = u\epsilon^{-m\xi + n\xi'}$, $m\,(a - b)^2 = B - Aa$, $n\,(a - b)^2 = B - Ab$, and $\xi = x - at$, $\xi' = x - bt$.

We may now transform equation (1) by writing wt^k for v, and assuming $k(k-1) = C'$.

$$\therefore\ D_t\Delta_t w + \frac{k}{t}(D_t + \Delta_t)\, w + cw = 0.$$

$$\therefore\ t\,(D_t\Delta_t + c)\, w + k\,(D_t + \Delta_t)\, w = 0 \quad \ldots\ldots\ldots\ldots(2).$$

Operate on this with $f(D_t\Delta_t)$ by the formula (3) of last Article.

$$\therefore\ \{tf(D_t\Delta_t) + (D_t + \Delta_t) f'(D_t\Delta_t)\}\,(D_t\Delta_t + c)\, w$$
$$+ k\,(D_t + \Delta_t) f(D_t\Delta_t)\, w = f(D_t\Delta_t)\, 0.$$

Now as the form of the function f is arbitrary we are at liberty to assume the sum of the terms after the first equal to zero.

$$\therefore\ f'(D_t\Delta_t)\,(D_t\Delta_t + c) + kf(D_t\Delta_t) = 0.$$

This is a linear equation of the first order.

$$\therefore\ f(D_t\Delta_t) = (D_t\Delta_t + c)^{-k};$$

$$\therefore\ t(D_t\Delta_t + c)^{1-k} w = (D_t\Delta_t + c)^{-k} 0 \quad \ldots\ldots\ldots\ldots(3);$$

$$\therefore\ w = (D_t\Delta_t + c)^{k-1} t^{-1} (D_t\Delta_t + c)^{-k} 0.$$

$$\therefore\ u = t^k \epsilon^{m\xi + n\xi'} . (D_t\Delta_t + c)^{k-1} t^{-1} (D_t\Delta_t + c)^{-k} 0.$$

This is the integral of the proposed equation; but it is obvious that it is only symbolical, except when k is an integer and $c = 0$, i.e. when

$$C(a-b)^2 + (B - Aa)(B - Ab) = 0.$$

Ex. 2. To integrate the equation given in the last example, when the conditions just stated are satisfied.

$$\therefore\ u = t^k \epsilon^{m\xi + n\xi'} . (D_t\Delta_t)^{k-1} \frac{1}{t} (S_t\Sigma_t)^k 0.$$

Now $$S_t^k \Sigma_t^k 0 = S_t^k 0 + \Sigma_t^k 0,$$

consequently u consists of two independent parts, one depending on $S_t^k 0$, and the other on $\Sigma_t^k 0$. We may seek them separately and similarly.

$$\frac{1}{t} S_t^k 0 = \frac{1}{t} (A_0 + A_1 t + A_2 t^2 + \dots + A_{k-1} t^{k-1})$$

$$= A_0 t^{-1} + A_1 + A_2 t + \dots + A_{k-1} t^{k-2},$$

A_0, A_1, A_2 ... being functions of ξ.

$$\therefore\ D_t^{k-1} . \frac{1}{t} S_t^k 0 = F(\xi) . t^{-k};$$

$$\therefore\ (D_t \Delta_t)^{k-1} \frac{1}{t} \Sigma_t^k 0 = \Delta_t^{k-1} . t^{-k} F(\xi).$$

Similarly $$(D_t \Delta_t)^{k-1} \frac{1}{t} \Sigma_t^k 0 = D_t^{k-1} t^{-k} f(\xi').$$

$$\therefore\ u = \epsilon^{m\xi + n\xi'} t^k \{D_t^{k-1} t^{-k} f(\xi') + \Delta_t^{k-1} t^{-k} F(\xi)\}.$$

This is the integral required; but it fails in the case of $k =$ a negative integer.

To find the integral in this case write $-k$ for k; then since $c = 0$, equations (2), (3) of example (1) become

$$t D_t \Delta_t w = k (D_t + \Delta_t) w,$$

and $$t (D_t \Delta_t)^{1+k} w = (D_t \Delta_t)^k 0 = 0;$$

$$\therefore\ w = S_t^{k+1} 0 + \Sigma_t^{k+1} 0.$$

The two parts of this integral are independent. We take them separately.

$$\therefore\ w = S_t^{k+1} 0 = F_0 \xi + t F_1 \xi + t^2 F_2 \xi + \dots + t^{k-1} F_{k-1} \xi + t^k F_k \xi.$$

Here we seem to have $k+1$ arbitrary functions of ξ, and there ought to be only one. The forms of the others must be determined by substituting this result in the equation $tD_t\Delta_t w = k\,(D_t+\Delta_t)\,w$. If this be done the following will be found to be the law of the functions,

$$F_{n+1}(\xi) = \frac{k-n}{2k-n}\cdot\frac{b-a}{n+1}\cdot d_\xi F_n(\xi).$$

In the same way we shall find

$$f_{n+1}(\xi) = \frac{k-n}{2k-n}\cdot\frac{a-b}{n+1}\cdot d_{\xi'} f_n(\xi');$$

$$\therefore\; u = \epsilon^{m\xi+n\xi'} t^{-k}\cdot\begin{cases} F_0\xi + tF_1\xi + t^2F_2\xi + \ldots + t^kF_k\xi \\ +f_0\xi' + tf_1\xi' + t^2f_2\xi' + \ldots + t^kf_k\xi'. \end{cases}$$

Ex. 3. To integrate the equation

$$d_t^2u + 2ad_td_xu + a^2d_x^2u + A\,(d_tu + ad_xu) + Bu = \frac{Cu}{t^2}.$$

In this example $D_t = d_t + ad_x$, and $\xi = x - at$;

$$\therefore\; D_t^2u + AD_tu + Bu = \frac{Cu}{t^2}.$$

For u write vt^m, and assume $m\,(m-1) = C$;

$$\therefore\; t\,(D_t^2 + AD_t + B)\,v + mD_tv = 0.$$

It will be convenient to assume

$$D_t^2 + AD_t + B = (D_t+\alpha)\,(D_t+\beta),$$

$$\text{now } mD_t = \frac{m}{\alpha-\beta}\cdot\{\alpha\,(D_t+\beta) - \beta\,(D_t+\alpha)\};$$

$$\therefore\; t\,(D_t+\alpha)\,(D_t+\beta)\,v + \frac{m\alpha}{\alpha-\beta}\,(D_t+\beta)\,v - \frac{m\beta}{\alpha-\beta}\,(D_t+\alpha)\,v = 0.$$

Differentiate this equation with $f(D_t)$;

$$\therefore\ (tfD_t + f'D_t)(D_t+\alpha)(D_t+\beta)v + \frac{m\alpha}{\alpha-\beta}(D_t+\beta)fD_t\,.\,v$$

$$-\frac{m\beta}{\alpha-\beta}(D_t+\alpha)fD_t\,.\,v = fD_t\,.\,0.$$

But because the form of f is arbitrary we are at liberty to assume, either

$$(D_t+\beta)fD_t - \frac{m\beta}{\alpha-\beta}.fD_t = 0, \text{ or, } (D_t+\alpha)fD_t + \frac{m\alpha}{\alpha-\beta}.fD_t = 0.$$

The former gives

$$fD_t = (D_t+\beta)^{\frac{m\beta}{\alpha-\beta}} \dots\dots\dots\dots\dots\dots (1)$$

and the latter gives

$$fD_t = (D_t+\alpha)^{\frac{m\alpha}{\beta-\alpha}} \dots\dots\dots\dots\dots\dots (2)$$

If we take the former, and for brevity write n for $\frac{m\beta}{\alpha-\beta}$, we have

$$tfD_t\,.\,(D_t+\alpha)(D_t+\beta)v + \frac{m\alpha}{\alpha-\beta}(D_t+\beta)fD_t\,.\,v = fD_t\,.\,0,$$

$$\text{or, } t(D_t+\alpha)\,.\,(D_t+\beta)^{n+1}v + \frac{m\alpha}{\alpha-\beta}\,.\,(D_t+\beta)^{n+1}v = (D_t+\beta)^n\,.\,0.$$

This being an equation of the first order with $(D_t+\beta)^{n+1}v$ as the dependent variable can be integrated as in the former part of this Essay.

REMARKS. It is to be remarked respecting this Example that the final integral will be only symbolical, unless n be an integer positive or negative. But as we might have used equation (2) instead of (1), we should succeed in the integration with that equation if $\frac{m\alpha}{\beta-\alpha}$ be an integer: and thus the only condition of success is that one of the two quantities $m\alpha$, $m\beta$ shall be a multiple of $\alpha-\beta$, where α, β are the two roots of the equation $\alpha^2 - A\alpha + B = 0$.

The method of integration will be similar if the right-hand member of the proposed equation be $C(ht+kx)^{-2}u$ instead of $Ct^{-2}u$.

Ex. 4. We come now to an example which has excited a good deal of attention, viz.

$$0 = d_t^2u + (a+b)\,d_td_xu + abd_x^2u + \frac{Ad_tu + Bd_xu}{ht+kx} + \frac{Cu}{(at+kx)^2}.$$

Here $D_t = d_t + ad_x$, $\Delta_t = d_t + bd_x$, $\xi = x - at$, $\xi' = x - bt$, $D_t(ht+kx) = h + ka$, $\Delta_t(ht+kx) = h + kb$.

In the proposed equation write $(ht+kx)^m v$ for u, and equate to zero the coefficient of v.

$$\therefore\ 0 = (h+ka)\,(h+kb)\,m\,(m-1) + (Ah+Bk)\,m + C,$$

and

$$0 = (ht+kx)(d_t^2u + \overline{a+b}\,.\,d_td_xu + abd_x^2u) + \{A + m(2h + \overline{a+b}\,.\,k)\}d_tv$$
$$+ \{B + m\,(2abh + \overline{a+b}\cdot h)\}\,d_xv\,;$$

Eliminate d_t and d_x,

$$\therefore\ 0 = (ht+kx)\,D_t\Delta_tv + \left\{\frac{B-Ab}{a-b} + m\,(h+bk)\right\}D_tv$$
$$+ \left\{\frac{B-Aa}{b-a} + m\,(h+ak)\right\}\Delta_tv.$$

For brevity represent the coefficients of D_tv and Δ_tv by

$$\alpha\,(h+bk),\ \text{ and }\ \beta\,(h+ak)\,;$$

$$\therefore\ 0 = (ht + kx)\, D_t\Delta_t v + \alpha\, (h + bk)\, D_t v + \beta\, (h + ak)\, \Delta_t v.$$

Operate on this with $f\,(\Delta_t)$,

$$\therefore\ f\,(\Delta_t)\, 0 = \{(ht + kx) f\Delta_t + (h + bk) f'\Delta_t\}\, D_t\Delta_t v$$
$$+ \alpha\, (h + bk)\, D_t f\Delta_t v + \beta\, (h + ak)\, \Delta_t f\Delta_t v.$$

Now as the form of f is arbitrary we may assume

$$\Delta_t f'\Delta_t + \alpha f\Delta_t = 0\,;$$
$$\therefore\ f\Delta_t = \Delta_t^{-\alpha} = \Sigma_t^{\alpha}.$$

Hence

$$f\Delta_t \,.\, 0 = (ht + kx) f\Delta_t \,.\, D_t\Delta_t v + \beta\, (h + ak)\, \Delta_t f\Delta_t v,$$

or $$\Sigma_t^{\alpha} 0 = (ht + kx)\, D_t \,.\, \Delta_t^{1-\alpha} v + \beta\, (h + ak)\, \Delta_t^{1-\alpha} v.$$

This is an equation of the first order with respect to $\Delta_t^{1-\alpha} v$ as the dependent variable, and may therefore be integrated as in Chap. II. Ex. 9.

Remark. The final integral will be only symbolical unless α be an integer; but as we might have operated with fD_t instead of $f\Delta_t$, we should succeed in this case if β be an integer; and consequently the proposed equation will be integrable in finite terms if either α or β, i.e. if either

$$m\,(a - b) + \frac{B - Ab}{h + kb} \text{ or } m\,(b - a) + \frac{B - Aa}{h + ka}$$

be a multiple of $a - b$. And as there are two values of m either of which we are at liberty to use, we shall be able to integrate the proposed equation in finite terms if one of the values of m will render one of the above quantities a multiple of $a - b$.

Ex. 5. To integrate

$$(ht + kx)\,(d_t^2 u + 2a d_t d_x u + a^2 d_x^2 u) + b\,(d_t u + a d_x u) + cu = 0.$$

In this example

$$D_t = d_t + a d_x,\quad \xi = x - at, \quad \text{and} \quad D_t(ht + kx) = h + ka;$$

$$\therefore\ (ht + kx) D_t^2 u + b D_t u + cu = 0.$$

Operate on this equation with $f(D_t)$;

$$\therefore\ \{(ht + kx) f(D_t) + (h + ka) f'(D_t)\} . D_t^2 u$$
$$+ b D_t . f(D_t) u + c f(D_t) u = f(D_t) . 0.$$

We are at liberty to assume

$$(h + ka) f'(D_t) . D_t^2 + (b D_t + c) f(D_t) = 0;$$

$$\therefore f(D_t) = D_t^{\frac{-b}{h+ka}} . \epsilon^{\frac{c D_t^{-1}}{h+ka}}$$

$$= S_t^{\frac{b}{h+ka}} . \epsilon^{\frac{c S_t}{h+ka}} \text{..................} (1);$$

$$\therefore\ (ht + kx) D_t^2 . f(D_t) u = f(D_t) . 0.$$

The occurrence of the exponential operative symbol in equation (1) shews that the proposed equation cannot be generally integrated in finite terms. Success would also require that b should be a multiple of $h + ka$.

Ex. 6. To integrate

$$d_t d_x u + A d_t u + B d_x u + Cu = \frac{C'}{t^2} . u.$$

Reduce the equation as in Art. 26, Case III., by assuming $u = v\epsilon^{-Ax - Bt}$;

$$\therefore\ d_t d_x v + (C - AB) v = \frac{C'}{t^2} . v \text{............} (1);$$

$$\therefore \frac{t^2 d_t}{(C - AB) t^2 - C'} . d_x v + v = 0.$$

Now assume a new variable τ such that

$$t^2 d_t \tau = (C - AB)\, t^2 - C';$$

$$\therefore\ \tau = (C - AB)\, t + \frac{C'}{t},$$

$$\text{and}\quad d_\tau d_x v + v = 0.$$

A form which we have seen cannot be integrated generally in finite terms. The proposed equation is therefore of a form which in general can only be integrated symbolically.

Ex. 7. To integrate

$$d_t d_x u + \frac{A d_t u + B d_x u}{ht + kx} + \frac{Cu}{(ht + kx)^2} = 0.$$

Write $(ht + kx)^m v$ for u, and equate to zero the coefficient of v;

$$\therefore\ 0 = hkm\,(m - 1) + (Ah + Bk)\, m + C,$$

and $\quad 0 = (ht + kx)\, d_t d_x v + (A + mk)\, d_t v + (B + mh)\, d_x v.$

Operate on this equation with $f(d_t)$;

$$\therefore\ \{(ht + kx)\, f d_t + h f' d_t\}\,.\, d_t d_x v + (A + mk)\, d_t f(d_t)\, v$$
$$+ (B + mh)\, d_x f(d_t)\, v = f(d_t)\,.\,0.$$

Now as the form of f is arbitrary we are at liberty to assume

$$h d_t f'(d_t) + (B + mh)\, f(d_t) = 0;$$

$$\therefore\ f(d_t) = (d_t)^{-m-\frac{B}{h}} = s_t^{\,m+\frac{B}{h}},$$

and $\quad (ht + kx)\, d_t f(d_t)\, d_x v + (A + mk)\, d_t f(d_t)\, v = f(d_t)\,.\,0,$

$$\therefore\ (ht + kx)\, d_x\,.\, d_t f(d_t)\, v + (A + mk)\, d_t f(d_t)\, v = f(d_t)\,.\,0.$$

This equation is of the first order with respect to $d_t f(d_t) v$ as the dependent variable, and it can be integrated by the general method given in the former part of this Essay. It is however essential to the finiteness of this integral that $\frac{B}{h}+m$ be an integer: but as we might have operated with $f(d_x)$ instead of $f(d_t)$, in which case the condition of finiteness would have depended on $\frac{A}{h}+m$ being an integer, it is evident that the proposed equation admits of integration in finite terms if either

$$\frac{A}{h}+m, \text{ or } \frac{B}{h}+m$$

be an integer. And as there are two values of m, either of which we are at liberty to use, the integral will be finite if either of the values of m will satisfy this condition.

Ex. 8. To integrate

$$d_t d_x u + A d_t u + B d_x u + ABu = \frac{Cu}{(ht+kx)^2}.$$

For u write $v\epsilon^{-Bt-Ax}$ as in Art. 24, Case III.;

$$\therefore\ d_t d_x v = \frac{Cv}{(ht+kx)^2}.$$

Now for v write $w(ht+kx)^m$, and assume

$$m(m-1)hk = C;$$

$$\therefore\ (ht+kx)\, d_t d_x w + mk d_t w + mh d_x w = 0.$$

Differentiate this equation with $f(d_t d_x)$;

$$\therefore\ \{(ht+kx) f(d_t d_x) + (hd_x + kd_t) f'(d_t d_x)\}\, d_t d_x w$$
$$+ m(kd_t + hd_x) f(d_t d_x)\, w = f(d_t d_x)\, 0.$$

We are at liberty to assume

$$f'(d_td_x)\,d_td_x + mf(d_td_x) = 0;$$

$$\therefore\ f(d_td_x) = (d_td_x)^{-m} = s_t^m s_x^m,$$

and
$$(ht+kx)\,f(d_td_x)\,d_td_xw = f(d_td_x)\,0;$$

$$\therefore\ (ht+kx)\,s_t^{m-1}s_x^{m-1}w = s_t^m s_x^m\,0 = s_t^m\,0 + s_x^m\,0.$$

Now $s_t^m\,0 = A_0 + A_1t + A_2t^2 + \dots + A_{m-1}t^{m-1},$

where A_0, A_1, A_2, … are arbitrary functions of x; but it will be more convenient to assume

$$s_t^m\,0 = A_0 + A_1(ht+kx) + A_2(ht+kx)^2 + \dots \\ \dots + A_{m-1}(ht+kx)^{m-1},$$

which we are at liberty to do because x is here a quasi-constant;

$$\therefore\ s_t^{m-1}s_x^{m-1}w = A_0(ht+kx)^{-1} + A_1 + \dots + A_{m-1}(ht+kx)^{m-2};$$

$$\therefore\ w = d_x^{m-1}(ht+kx)^{-m}F(x).$$

Similarly we have

$$w = d_t^{m-1}(ht+kx)^{-m}f(t).$$

Hence the complete integral is

$$w = d_t^{m-1}(ht+kx)^{-m}f(t) + d_x^{m-1}(ht+kx)^{-m}F(x).$$

If m be a negative integer, write $-m$ for it; then

$$f(d_td_x) = d_t^m d_x^m;$$

$$\therefore\ (kt+kx)\,d_t^{m+1}d_x^{m+1}w = d_t^m d_x^m\,0 = 0;$$

$$\therefore\ w = s_t^{m+1}s_x^{m+1}\,0 = s_t^{m+1}\,0 + s_x^{m+1}\,0 \\ = F_0x + (ht+kx)F_1x + \dots + (ht+kx)^{m-1}F_{m-1}x \\ + f_0t + (ht+kx)f_1t + \dots + (ht+kx)^{m-1}f_{m-1}t.$$

The laws of the arbitrary functions must be determined as in Art. 40, Ex. 2, by substituting this result in the equation from which they were derived, viz.

$$(ht + kx)\, d_t d_x w = mkd_t w + mhd_x w.$$

If this be done we find

$$F_{n+1}x = \frac{n-m}{(2m-n)(n+1)} \cdot \frac{1}{k} \cdot F_n' x,$$

$$f_{n+1}t = \frac{n-m}{(2m-n)(n+1)} \cdot \frac{1}{h} \cdot f_n' t.$$

Ex. 9. The equation

$$t\phi(d_t)\, u + A\phi'(d_t)\, u = 0$$

can be deprived of its last term, and integrated if A be an integer.

Differentiate it with $f(d_t)$;

$$\therefore\ (tfd_t + f'd_t)\, \phi(d_t)\, u + Af(d_t)\, \phi'(d_t)\, u = f(d_t)\, 0.$$

We are at liberty to assume

$$\phi(d_t) f'd_t + A\, \phi'(d_t) fd_t = 0;$$

$$\therefore\ fd_t = (\phi d_t)^{-A},$$

and $$t\, (\phi d_t)^{1-A}\, u = (\phi d_t)^{-A}\, 0;$$

$$\therefore\ u = (\phi d_t)^{A-1}\, t^{-1}\, (\phi d_t)^{-A}\, 0.$$

Ex. 10. The equation

$$(ht + kx)\, \phi(D_t)\, u + A\, (h + ka)\, \phi'(D_t)\, u = 0,$$

where $D_t x = a$, if treated in the same manner, will give

$$u = (\phi D_t)^{A-1}\, (ht + kx)^{-1}\, \phi\, (D_t)^{-A}\, 0.$$

41. If we have a linear differential equation, the coefficients of which are of the form $A_n t^n + A_{n-1} t^{n-1} + \dots$ we may arrange the terms of the equation according to the powers of t. It will then take the form

$$\phi_0(d_t)\,u + t\phi_1(d_t)\,u + t^2\phi_2(d_t)\,u + \dots = 0.$$

We shall shew how every equation of the kind here proposed may be solved, at least symbolically. And this we shall do by shewing how the equation can be reduced to another of one dimension lower in t: and as this process may be repeated on the reduced equation, we shall at length arrive at an equation of the form

$$t\psi_1(d_t)\,u = \psi_0\,0,$$

from which u will be known. We shall not attempt to prove this in a general way, in order to avoid complexity, but a few examples will be given which will render the general proof unnecessary.

Ex. 11. To reduce to

$$t\phi_1(d_t)\,u + \phi_0(d_t)\,u = 0.$$

Operate on this question with $f(d_t)$;

$$\therefore\ (tfd_t + f'd_t)\phi_1(d_t)\,u + fd_t\,.\,\phi_0(d_t)\,u = fd_t\,.\,0.$$

As the form of f is arbitrary we are at liberty to assume

$$f'(d_t)\,\phi_1(d_t) + f(d_t)\,\phi_0(d_t) = 0 \;\dots\dots\dots\dots (1);$$

$$\therefore\ tf(d_t)\,\phi_1(d_t)\,u = f(d_t)\,0\,;$$

$$\therefore\ u = (fd_t\,.\,\phi d_t)^{-1}\,t^{-1}f(d_t)\,0.$$

Equation (1) will give the form of $f(d_t)$, and that of $\phi_1(d_t)$ is given in the proposed equation: and consequently u is symbolically known from the last equation.

Ex. 12. To reduce

$$t^2\phi_2(d_t)u + t\phi_1(d_t)u + \phi_0(d_t)u = 0,$$

one degree lower in the powers of t.

Differentiate with $f(d_t)$;

$$\therefore\ (t^2fd_t + 2tf'd_t + f''d_t)\,\phi_2(d_t)\,u + (tfd_t + f'd_t)\,\phi_1(d_t)\,.\,u$$
$$+ f(d_t)\,.\,\phi_0(d_t)\,u = f(d_t)\,0.$$

Equate to zero the terms on the left-hand side which do not contain t, and divide the remaining portion by t;

$$\therefore\ \phi_2(d_t)\,.\,f''(d_t) + \phi_1(d_t)\,.\,f'(d_t) + \phi_0(d_t)\,.\,f(d_t) = 0,$$

and

$$tf(d_t)\,\phi_2(d_t)\,u + \{2f'(d_t)\,.\,\phi_2(d_t) + f(d_t)\,\phi_1(d_t)\}\,u = t^{-1}f(d_t)\,0.$$

The former of these being of the second order will give two forms of $f(d_t)$, either of which may be used: and the latter equation falls under the preceding example.

These two examples are sufficient to exhibit the principle of the method, and to shew that any equation arranged according to powers of t can be integrated, symbolically at least, if the successive equations for finding the operative factorial $f(d_t)$ can be solved.

It is hardly necessary to say that we may write $ht + kx$ for t, and D_t for d_t; and a similar proposition will hold good respecting an equation of the form

$$\phi_0(D_t)\,u + (ht + kx)\,\phi_1(D_t)\,u + (ht + kx)^2\,\phi_2(D_t)u + \ldots = 0,$$

if $D_tx =$ constant, or quasi-constant.

Ex. 13. To integrate

$$t\,(d_t + \alpha)\,(d_t + \beta)\,u = A\,(d_t + \alpha)\,u + B\,(d_t + \beta)\,u.$$

Differentiate with $f(d_t)$,

$$\therefore\ (tfd_t + f'd_t)(d_t + \alpha)(d_t + \beta)\,u = Af(d_t)(d_t + \alpha)\,u$$
$$+ Bf(d_t)(d_t + \beta)\,u + f(d_t)\,0.$$

We are at liberty to assume $f'(d_t)(d_t + \alpha)(d_t + \beta)\,u$ equal to either of the terms of the right-hand member which contain u;

$$\therefore\ \frac{f'd_t}{fd_t} = \frac{A}{d_t + \beta}, \text{ or } = \frac{B}{d_t + \alpha};$$

$$\therefore\ f(d_t) = (d_t + \beta)^A, \text{ or } (d_t + \alpha)^B.$$

If now either A or B be an integer, we may take the corresponding form for $f(d_t)$. Suppose A an integer;

$$\therefore\ f(d_t) = (d_t + \beta)^A,$$

and $\quad t(d_t + \alpha)(d_t + \beta)^{A+1}\,u = B(d_t + \beta)^{A+1}\,u + (d_t + \beta)^A\,0.$

If we write v for $(d_t + \beta)^{A+1}u$, we have

$$t(d_t + \alpha)\,v = Bv + (d_t + \beta)^A\,0,$$

which being an equation of the first order can be integrated as in the early part of this Essay. The proposed equation can therefore be integrated in finite terms if either A or B be an integer.

CHAPTER VIII.

θ OR $\log t$ AS LEADING INDEPENDENT VARIABLE.

42. **There** is an important class of equations which seem to be most easily integrated by the use of θ $(= \log t)$ as leading independent variable. A few of the requisite formulæ are added.

Since $t = \epsilon^\theta$; $\therefore\ d_\theta t = t$ ....(1).

Again, since $d_\theta u = d_t u \,.\, d_\theta t = t d_t u$;

$$\therefore\ d_\theta = t d_t \text{ .... } (2).$$

Again; $d_\theta \,.\, t^{-n} u = t^{-n} (d_\theta - n)\, u$;

$$\therefore\ t^n d_\theta t^{-n} = d_\theta - n \text{ .... } (3);$$

$$\therefore\ t^{n+1} d_t t^{-n} = d_\theta - n \text{ .... } (4) \text{ from } (2).$$

In (4) write 0, 1, 2, 3, ... $(n-1)$ for n; multiply the results together, and reverse them;

$$\begin{aligned}\therefore\ d_\theta (d_\theta - 1)(d_\theta - 2) \ldots (d_\theta - n + 1) & \\ &= (d_\theta - n + 1)(d_\theta - n + 2) \ldots (d_\theta - 1)\, d_\theta \\ &= t^n d_t t^{1-n} \,.\, t^{n-1} d_t t^{2-n} \ldots\ldots t^2 d_t t^{-1} \,.\, t d_t \\ &= t^n \,.\, d_t^n \text{ .... } (5).\end{aligned}$$

43. The following formulæ of operations are useful abbreviations.

Since $$(d_\theta + k) = t^{-k} d_\theta t^k;$$

$$\therefore (d_\theta + k)^n = t^{-k} d_\theta t^k \,.\, t^{-k} d_\theta t^k \,.\, t^{-k} d_\theta t^k \ldots \text{ to } n \text{ terms,}$$
$$= t^{-k} d_\theta^n t^k \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$
$$\therefore f(d_\theta + k) = t^{-k} f(d_\theta) t^k \ldots\ldots\ldots\ldots\ldots(2);$$
$$\therefore f(d_\theta + k) \,.\, f(d_\theta + 2k) \ldots f(d_\theta + nk)$$
$$= t^{-k} f(d_\theta)\, t^k \,.\, t^{-2k} f(d_\theta)\, t^{2k} \ldots t^{-nk} f(d_\theta) t^{nk}$$
$$= (t^{-k} f d_\theta)^n \,.\, t^{nk} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Since $$f(d_\theta + h + k) = t^{-h} f(d_\theta + k)\, t^h,$$
and $$f(d_\theta + h + 2k) = t^{-h} f(d_\theta + 2k)\, t^h,$$
$$\ldots\ldots = \ldots\ldots$$
$$\therefore f(d_\theta + h + k) \,.\, f(d_\theta + h + 2k) \ldots f(d_\theta + h + nk)$$
$$= t^{-h} f(d_\theta + k) f(d_\theta + 2k) \ldots f(d_\theta + nk)\, t^h$$
$$= t^{-h} (t^{-k} f d_\theta)^n \,.\, t^{nk+h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

It is very important to notice that the formulæ numbered (1), (2), (3), (4) in this, and (1), (2), (3), (4), (5) in the preceding article are commutative with d_θ and with each other, because their equals in terms of d_θ are so.

44. By formula (2) of the last article
$$f(d_\theta + h) = t^{-h} f(d_\theta)\, t^h;$$
$$\therefore (d_\theta + h)(d_\theta + h - 1)(d_\theta + h - 2) \ldots\ldots (d_\theta + h - n + 1)$$
$$= t^{-h} d_\theta \,.\, (d_\theta - 1) \,.\, (d_\theta - 2) \ldots\ldots (d_\theta - n + 1)\, t^h$$
$$= t^{-h} \,.\, t^n d_t^n \,.\, t^h \ldots\ldots\ldots \text{(Art. 42, 5)}.$$

By this formula we are able to express the product of a series of factorials of the form $(d_\theta + \alpha)$ $(d_\theta + \beta)$ $(d_\theta + \gamma) \ldots$ $(d_\theta + \mu)$ in terms of t and d_t, on the *one* condition that α, β,

$\gamma \ldots \mu$ can be so arranged as to form an Arithmetical progression *whose common difference is unity.*

And it is in consequence of this ability to pass from d_θ to d_t with such facility when such a series can be formed out of α, β, $\gamma \ldots \mu$, that in the following articles our chief object will be to change the values of α, β, $\gamma \ldots \mu$ or some of them, if necessary, so as to produce such a series when possible.

45. THEOREM. If we have an equation of the form $f(d_\theta) . \phi(d_\theta) u + Ct^k u = 0$, then by writing vt^{-a} for u, the subjects of f and ϕ will be *both* reduced by a.

For $$f(d_\theta)\, \phi(d_\theta)\, t^{-a} v + Ct^k . t^{-a} v = 0\,;$$

$$\therefore\ t^a f(d_\theta)\, \phi(d_\theta)\, t^{-a} . v + Ct^k v = 0\,;$$

$$\therefore\ f(d_\theta - a)\, \phi(d_\theta - a)\, v + Ct^k v = 0 \quad \text{(Art. 43, 2)}.$$

Hence the following Rule:—To pass from the proposed equation to this, or from this to the proposed equation, nothing more is necessary than to use the equation

$$u = vt^{-a}, \text{ or } v = ut^a.$$

46. By the last Article we are able to reduce the subjects of the operative factorials when they are to be affected alike; but if we wish to affect one of them only we must proceed differently.

Given the equation $f(d_\theta)\, \phi(d_\theta)\, u + Ct^k u = 0$, to increase or diminish the subject of f, without altering that of ϕ.

Commutate f and ϕ, and differentiate the equation with $t^{-h} (t^{-k} f d_\theta)^n t^{nk+h}$ (see Art. 43, 4); and assume $nk + h = -k$;

$$\therefore\ \phi(d_\theta)\, t^{-h} (t^{-k} f d_\theta)^n t^{-k} f(d_\theta)\, u + Ct^{-h} (t^{-k} f d_\theta)^n u = 0.$$

Now for $(t^{-k} f d_\theta)^n u$ write $(t^{-k})^n v$.

$$\therefore\ \phi\,(d_\theta)\,t^{nk} f\,(d_\theta)\,t^{-nk} v + Ct^k v = 0\,;$$

$$\therefore\ \phi\,(d_\theta)\,f\,(d_\theta - nk)\,v + Ct^k v = 0 \ldots\ldots \text{Art. 43, 2.}$$

THEOREM. Hence if we desire to pass from the equation,

$$f\,(d_\theta)\,\phi\,(d_\theta)\,u + Ct^k u = 0,\ \text{ to }\ f\,(d_\theta - nk)\,\phi\,(d_\theta) v + Ct^k v = 0,$$

or, from the latter to the former, we must employ the equation

$$(t^{-k} f d_\theta)^n u = (t^{-k})^n v.$$

47. Though we have thus succeeded in diminishing or increasing the subject of one operative function without affecting the other, it is to be noted that we can only increase or diminish a subject by integer multiples of k.

48. If we have three factorials $f\,(d_\theta)$, $F(d_\theta)$, $\phi\,(d_\theta)$, we can change the subject of f by nk; and then that of F by $n'k$; and then that of ϕ by $n''k$; and so on for any number of operative factorials; and to accomplish this we have merely to employ in succession the equations

$$(t^{-k} f d_\theta)^n u = (t^{-k})^n v,$$
$$(t^{-k} f d_\theta)^{n'} v = (t^{-k})^{n'} w,$$
$$\&\text{c.} = \&\text{c.}$$

Thus the result of the preceding articles is, that if we have an equation of the form

$$f\,(d_\theta)\,.\,F\,(d_\theta)\,.\,\phi\,(d_\theta)\,w + Ct^k w = 0,$$

we can affect the subjects of f, F, ϕ in two ways:—1st, we can alter them all alike by any proposed quantity; and 2nd, we can alter any one or more of them by multiples of k.

49. In practice a somewhat more convenient form of the preceding equation is the following.

THEOREM. We can pass from one of the two following equations, viz.

$$f(d_\theta+\alpha)\phi(d_\theta)\,u+Ct^k u=0, \text{and } f(d_\theta+\alpha-nk)\,\phi(d_\theta)v+Ct^k v=0,$$

to the other by means of the equation,

$$(t^{-k}fd_\theta)^n.\,ut^\alpha=(t^{-k})^n.\,vt^\alpha.$$

For since $f(d_\theta+\alpha)=t^{-\alpha}f(d_\theta)\,t^\alpha$......(Art. 43, 2);

$$\therefore\ f(d_\theta)\,t^\alpha\phi(d_\theta)\,u+Ct^k.\,ut^\alpha=0;$$

$$\therefore\ f(d_\theta)\,t^\alpha\phi(d_\theta)\,t^{-\alpha}.\,ut^\alpha+Ct^k.\,ut^\alpha=0;$$

$$\therefore\ f(d_\theta)\,\phi(d_\theta-\alpha).\,ut^\alpha+Ct^k.\,ut^\alpha=0.$$

Let us now assume (in order to reduce the subject of f by nk) in this equation,

$$(t^{-k}fd_\theta)^n.\,ut^\alpha=(t^{-k})^n.\,vt^\alpha \text{ (Article 46).}$$

$$\therefore\ f(d_\theta-nk).\,\phi(d_\theta-\alpha)\,vt^\alpha+Ct^k.\,vt^\alpha=0.$$

Divide by t^α;

$$\therefore\ t^{-\alpha}f(d_\theta-nk)\,\phi(d_\theta-\alpha)\,t^\alpha v+Ct^k v=0;$$

$$\therefore\ f(d_\theta+\alpha-nk)\,\phi(d_\theta).\,v+Ct^k v=0. \text{ (Art. 43, 2.)}$$

50. From this it is evident that we may pass from one of the following equations to the other,

$$f(d_\theta+\alpha)\,F(d_\theta+\beta)\,\phi(d_\theta)\,u+Ct^k u=0,$$

$$\text{and } f(d_\theta+\alpha-nk)\,F(d_\theta+\beta-n'k)\,\phi(d_\theta)\,w+Ct^k w=0,$$

by assuming, in succession, the equations

$$(t^{-k}fd_\theta)^n.\,ut^\alpha=(t^{-k})^n.\,vt^\alpha,$$

$$(t^{-k}fd_\theta)^{n'}.\,vt^\beta=(t^{-k})^{n'}.\,wt^\beta.$$

Ex. 1. To reduce $(d_\theta + a)(d_\theta + 5)u + Ct^k u = 0$ to a simpler form in d_t.

Reduce the first factor by nk,

$$\therefore\ (d_\theta + a - nk)(d_\theta + 5)v + Ct^k v = 0 \ldots\ldots\ldots(1),$$

and $$(t^{-k}d_\theta)^n . ut^a = (t^{-k})^n . vt^a \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now we have to form, if possible, of the two operative factors of equation (1) two adjacent terms of an Arithmetic Series whose common difference is unity. (Art. 44.)

$$\therefore\ a - nk - 5 = \pm 1.$$

We have therefore to consider the two cases, $a = nk + 6$ and $a = nk + 4$.

Case 1. When $a = nk + 6$, equation (1) becomes

$$(d_\theta + 6)(d_\theta + 5)v + Ct^k v = 0;$$

$$\therefore\ d_\theta(d_\theta - 1) . vt^6 + Ct^k . vt^6 = 0 \quad \text{(Art. 43, 2)};$$

$$\therefore\ d_t^2 . vt^6 + Ct^{k-2} . vt^6 = 0 \ldots\ldots\ldots(3).$$

Case 2. When $a = nk + 4$, equation (1) becomes

$$(d_\theta + 4)(d_\theta + 5)v + Ct^k v = 0,$$

$$\therefore\ (d_\theta - 1) d_\theta . vt^5 + Ct^k . vt^5 = 0, \quad \text{(Art. 43, 2)},$$

$$\therefore\ d_t^2 . vt^5 + Ct^{k-2} . vt^5 = 0 \quad \ldots\ldots\ldots\ldots(4).$$

The reduced equations (3) and (4) are of the same form, and if that form can be integrated then the value of u may be found from (2).

In the *first* case, if V be the integral of the equation

$$d_t^2 V + Ct^{k-2} V = 0,$$

we find from equation (2)

$$u = t^{-nk-6}(s_t t^{k-1})^n V.$$

In the *second* case,

$$u = t^{-nk-4}(s_t t^{k-1})^n Vt.$$

Ex. 2. To integrate the equation

$$d_t^2 u + \frac{A}{t} \,.\, d_t u + Ct^{k-2} u = 0.$$

Multiply by t^2 and introduce d_θ;

$$\therefore\ (d_\theta + A - 1)\, d_\theta u + Ct^k u = 0.$$

Diminish the first factor by nk (Art. 49);

$$\therefore\ (t^{-k} d_\theta)^n \,.\, ut^{A-1} = (t^{-k})^n \,.\, vt^{A-1} \ \ldots\ldots\ldots\ldots\ldots(1)$$

and $$(d_\theta + A - nk - 1)\, d_\theta v + Ct^k v = 0.$$

Hence we must have $A - nk - 1 = \pm 1$; there are therefore two cases to consider.

1st, when $A = nk$;

$$(d_\theta - 1)\, d_\theta v + Ct^k v = 0;$$

$$\therefore\ d_t^2 v + Ct^{k-2} v = 0 \text{ (Art. 42, 5) } \ldots\ldots(2),$$

2nd, when $A = nk + 2$;

$$\therefore\ (d_\theta + 1)\, d_\theta v + Ct^k v = 0;$$

$$\therefore\ d_\theta (d_\theta - 1) \,.\, vt + Ct^k \,.\, vt = 0 \text{ (Art. 45)};$$

$$\therefore\ d_t^2 \,.\, vt + Ct^{k-2} \,.\, vt = 0 \ \ldots\ldots\ldots\ldots(3).$$

The value of u will be known from equation (1), if the equation

$$d_t^2\, V + Ct^{k-2} V = 0$$

can be integrated.

In the *first* case

$$u = t^{1-nk}(s_t t^{k-1})^n \cdot \frac{V}{t}.$$

In the *second* case

$$u = t^{-1-nt}(s_t t^{k-1})^n \cdot Vt.$$

Ex. 3. To find for what values of k the equation $d_t^2 u + Ct^{k-2}u = 0$ is integral in finite terms.

Change the independent variable from t to τ, where $\tau = t^n$.

$$\therefore\ \tau d_\tau^2 u + \frac{n-1}{n} \cdot d_\tau u + \frac{C}{n^2} \cdot t^{k-n} u = 0.$$

As n is at our disposal we may assume $t^{k-n} = \tau$, which gives $2n = k$.

$$\therefore\ \tau\left(d_\tau^2 + \frac{C}{n^2}\right)u + \frac{k-2}{2k}(2d_\tau)u = 0.$$

Now on referring to Article 40, Example 9, it will be seen that this exactly corresponds with the equation there integrated, $\phi(d_t)$ there representing $d_\tau^2 + \frac{C}{n^2}$ here. The only condition of integrability in finite terms is that A in that example, which represents $\frac{k-2}{2k}$ in this example, shall be an integer;

$$\therefore\ \frac{k-2}{2k} = \pm\lambda, \text{ an integer};$$

$$\therefore\ k = \frac{2}{1 \mp 2\lambda}.$$

Hence if k be a fraction, the numerator of which is 2, and the denominator of which is any positive or negative odd integer, the proposed equation will be integrable; and its integral will be

$$u = \left(d_\tau^2 + \frac{C}{n^2}\right)^{\pm\lambda-1} t^{-1} \left(d_\tau^2 + \frac{C}{n^2}\right)^{\mp\lambda} . 0.$$

REMARK. C may be a function of d_x.

Ex. 4. To take away the second term of the equation $d_t^m u + \frac{A}{t} . d_t^{m-1} u + Ct^{k-m} u = 0.$

Multiply by t^m and introduce d_θ;

$$\therefore\ t^{m-1}(d_\theta + A) d_t^{m-1} u + Ct^k u = 0;$$

$$\therefore\ t^{m-1}(d_\theta + A)\, t^{1-m} . t^{m-1} d_t^{m-1} u + Ct^k u = 0;$$

$$\therefore\ (d_\theta + A - m + 1)\, \phi(d_\theta)\, u + Ct^k u = 0 \text{ (Art. 42, 3)...(1)}$$

where $\phi(d_\theta) = t^{m-1} d_t^{m-1} = d_\theta(d_\theta - 1)\ldots(d_\theta - m + 2)$. (Art. 42, 5).

Diminish the first factor of (1) by nk;

$$\therefore\ (d_\theta + A - nk - m + 1)\, \phi(d_\theta)\, v + Ct^k v = 0 \text{ ......(2),}$$

and $$(t^{-k} d_\theta)^n . u t^{A-m+1} = (t^{-k})^n . v t^{A-m+1} \text{............(3).}$$

On looking at the factorial form of $\phi(d_\theta)$ it will be seen at once (Art. 44) that we may only place the first factorial of (2) either before $\phi(d_\theta)$, or after it. There are therefore two cases for consideration.

1st. Let $A = nk$; then $d_\theta + A - nk - m + 1$ comes after $\phi(d_\theta)$.

$$\therefore\ \phi(d_\theta)\,(d_\theta - m + 1)\, v + Ct^k v = 0;$$

$$\therefore\ d_t^m v + Ct^{k-m} v = 0 \text{.................(4).}$$

2nd. Let $A = nk + m$; then $d_\theta + A - nk - m + 1$ comes before $\phi(d_\theta)$.

$$\therefore\ (d_\theta + 1)\,\phi(d_\theta)\,v + Ct^k v = 0;$$

$$\therefore\ d_\theta\,.\,\phi(d_\theta - 1)\,.\,vt + Ct^k\,.\,vt = 0 \text{ (Art. 45)};$$

$$\therefore\ d_t^m\,.\,vt + Ct^{k-m}\,.\,vt = 0 \ldots\ldots\ldots\ldots(5).$$

These equations (4) and (5) are of the same form as the proposed equation deprived of its second term; and if this form can be integrated, v will be known, and then u from (3).

Ex. 5. To deprive

$$d_t^m u + \frac{A+B+1}{t}\,.\,d_t^{m-1}u + \frac{AB}{t^2}\,d_t^{m-2}u + Ct^{k-m}u = 0$$

of its second and third terms.

Multiply by t^m and introduce d_θ;

$$\therefore\ (d_\theta + A - m + 2)\,(d_\theta + B - m + 2)\,\phi(d_\theta)\,u + Ct^k u = 0 \ldots(1),$$

where $\qquad \phi(d_\theta) = d_\theta\,(d_\theta - 1)\ldots(d_\theta - m + 3).$

Diminish the first factor of (1) by nk, and the second by $n'k$ (Art. 50),

$$\therefore\ (d_\theta + A - nk - m + 2)\,(d_\theta + B - n'k - m + 2)\,\phi(d_\theta)\,w$$
$$+ Ct^k w = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and
$$\left.\begin{aligned}(t^{-k}d_\theta)^n\,.\,ut^{A-m+2} &= (t^{-k})^n\,.\,vt^{A-m+2}\\ (t^{-k}d_\theta)^{n'}\,.\,vt^{B-m+2} &= (t^{-k})^{n'}\,.\,wt^{B-m+2}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(3).$$

Now there are only three ways in which we can dispose, in accordance with Art. 44, the first two factorials of (2): viz. we can place them (1st) *after* $\phi(d_\theta)$; (2nd) one *before* and the other *after* $\phi(d_\theta)$; (3rd) both *before* $\phi(d_\theta)$. There are therefore these three cases to be taken into consideration.

(1). Let $A = nk$, $B = n'k - 1$. In this case both are after $\phi(d_\theta)$.

$$\therefore\ A + B + 1 = (n + n')\,k, \text{ and } AB = nk\,(n'k - 1),$$

and $\phi(d_\theta)(d_\theta - m + 2)(d_\theta - m + 1)\,w + Ct^k\,w = 0$;

$$\therefore\ d_t^m\,.\,w + Ct^{k-m}\,.\,w = 0 \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

(2). Let $A = nk$, $B = n'k + m - 1$. In this case one is before and one after $\phi(d_\theta)$.

$\therefore\ A + B + 1 = (n + n')\,k + m$, and $AB = nk\,(n'k + m - 1)$;

and $(d_\theta + 1)\,\phi(d_\theta)\,(d_\theta - m + 2)\,w + Ct^k w = 0$;

$$\therefore\ d_\theta\phi\,(d_\theta - 1)\,(d_\theta - m + 1)\,.\,wt + Ct^k\,.\,wt = 0;$$

$$\therefore\ d_t^m\,.\,wt + Ct^{k-m}\,.\,wt = 0 \ldots\ldots\ldots\ldots\ldots(5).$$

(3). Let $A = nk + m$, $B = n'k + m - 1$. In this case both are before $\phi(d_\theta)$.

$\therefore\ A + B + 1 = (n + n')\,k + 2m$, and $AB = (nk + m)(n'k + m - 1)$;

and $(d_\theta + 2)\,(d_\theta + 1)\,\phi(d_\theta)\,w + Ct^k\,w = 0$;

$$\therefore\ d_\theta\,(d_\theta - 1)\,\phi\,(d_\theta - 2)\,.\,wt^2 + Ct^k\,.\,wt^2 = 0;$$

$$\therefore\ d_t^m\,.\,wt^2 + Ct^{k-m}\,.\,wt^2 = 0 \ldots\ldots\ldots\ldots\ldots(6).$$

The equations (4), (5), (6) are of the same form as the proposed equation deprived of its second and third terms. If this form ($d_t^2 V + Ct^{k-m} V = 0$) can be integrated, the value of u will be given by equations (3).

Ex. 6. To deprive the equation

$$d_t^m u + \frac{2A + 1}{t}\,.\,d_t^{m-1} u + \frac{A^2}{t^2}\,.\,d_t^{m-2} u + Ct^{k-m} u = 0$$

of its second and third terms.

This is a particular case of the last example, and treated in the same way it gives

$$(d_\theta + A - m + 2)^2\phi\,(d_\theta)\,u + Ct^k u = 0 \ldots\ldots\ldots\ldots(1).$$

The factors before $\phi(d_\theta)$ being *equal*, we cannot place them *together* so as to constitute two adjacent terms of an arithmetic series whose common difference is unity. The only course open to us is, to diminish one by nk, and the other by $n'k$, and then place one *before* $\phi(d_\theta)$ and the other *after* $\phi(d_\theta)$, as in the second case of last example.

$$\therefore\ A = nk + m - 1 = n'k \ldots\ldots\ldots\ldots\ldots(2),$$

and
$$(d_\theta + 1)\,\phi(d_\theta)\,(d_\theta - m + 2)\,w + Ct^k w = 0\,;$$

$$\therefore\ d_t^m\,.\,wt + Ct^{k-m}\,.\,wt = 0 \ldots\ldots\ldots\ldots\ldots(3).$$

From (2) we learn that A and $m-1$ must be multiples of k.

Ex. 7. To deprive

$$d_t^m u - \frac{A\,(A+1)}{t^2}\,.\,d_t^{m-2} u + Ct^{k-m} u = 0$$

of its second term.

This, like the last example, is a particular case of Ex. 5, and treated in the same way it gives (since $B = -A-1$);

$$(d_\theta + A - m + 2)\,(d_\theta - A - m + 1)\,\phi(d_\theta)\,u + Ct^k u = 0.$$

Corresponding to the three cases of Ex. 5, we have

(1). Let $A = nk$, which gives $n' = -n$.

$$\therefore\ A\,(A+1) = nk\,(nk+1)\,;$$

and
$$d_t^m w + Ct^{k-m} w = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

It is a condition in this case that A shall be a multiple of k.

(2). Let $A = nk$, which gives $(n + n')\,k + m = 0$;

$$\therefore\ A\,(A+1) = nk\,(nk+1),$$

and $$d_t^m \,.\, wt + Ct^{k-m} \,.\, wt = 0 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

It is a condition in this case that m and A shall be multiples of k.

(3). Let $A = nk + m$, which gives $(n + n')\,k + 2m = 0$;

$$\therefore\ A\,(A+1) = (nk+m)\ (nk+m+1)\,;$$

and $$d_t^m \,.\, wt^2 = Ct^{k-m} \,.\, wt^2 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It is a condition in this case that $A - m$, and $2m$ shall be multiples of k.

In these three cases if the proposed equation when deprived of its middle term can be integrated, then the integral of the equation with its middle term can be deduced from that integral by means of the equations (3) of Example (5).

51. The method of reduction employed in the preceding examples being general enables us to generalise the results. Thus if we have an equation of the form

$$d_t^m u + \frac{A'}{t} \,.\, d_t^{m-1} u + \frac{B'}{t^2} \,.\, d_t^{m-2} u + \ldots + \frac{L'}{t^\lambda} \,.\, d_t^{m-\lambda} u + Gt^{k-m} u = 0,$$

we must multiply this by t^m, introduce d_θ, and cast the first part of the resulting equation into the form of operative factorials. It will then have the form

$$(d_\theta + A - m + \lambda)\,(d_\theta + B - m + \lambda) \ldots (d_\theta + L - m + \lambda)\ \phi\,(d_\theta)\, u$$
$$+ Gt^k u = 0,$$

where $$\phi\,(d_\theta) = d_\theta\,(d_\theta - 1) \ldots (d_\theta - m + \lambda + 1).$$

All the symbolic factors preceding $\phi(d_\theta)$ in this equation are to be increased or diminished by multiples of k; and they are then to be arranged in every possible way with $\phi(d_\theta)$ in which they can form with $\phi(d_\theta)$ a series of m factorials in arithmetic progression whose common difference is unity. There are, speaking generally, and subject to conditions as in the preceding examples, $\lambda+1$ ways of doing this: and in any of the cases if μ be the number of factorials which stand *before* $\phi(d_\theta)$, then $\lambda-\mu$ will be the number that come *after* $\phi(d_\theta)$, and the reduced equation will be

$$d_t^m . Vt^\mu + Gt^{k-m} . Vt^\mu = 0.$$

If this equation can be integrated, then the integral of the proposed equation can be at once deduced from it.

52. The principal object of this Chapter has been the reduction of equations to integrable forms by depriving them of some of their terms. The symbol d_t has been used throughout and d_x, d_y, d_z ... have nowhere appeared, so that it would seem as though this Chapter has no connexion with the subject of partial differential equations. It is therefore necessary to point out that the articles of this Chapter are all true when D_t is used for d_t, and S_t for s_t. And not only so, but that even when d_t is used, the coefficient which we have denoted by C may be a function of d_x, d_y, d_z ..., without affecting the truth of the reductions of the proposed equations. Also when D_t is substituted for d_t, then C may be a quasi-constant.

Also since $D_t(ht+kx) = h+ka$ when $D_t x = a$, the propositions admit of our writing $ht+kx$ instead of t when D_t is used. The methods of reduction will be just the same, and the results similar in form. And the quantity a may be a quasi-constant.

Thus $$(ht+kx)^n D_\theta (ht+kx)^{-n} = D_\theta - n(h+ka)$$
$$= t^{n(h+ka)} D_\theta t^{-n(h+ka)}.$$

In this case the only effect of using $ht+kx$ for t is to substitute $n(h+ka)$ for n. It will be found that only constants and not forms will be affected by the change of writing $ht+kx$ for t.

CHAPTER IX.

MISCELLANEOUS EXAMPLES.

53. THE design of this Chapter is the illustration of various modes of applying the principles and methods of former Chapters.

Ex. 1. To integrate

$$tD_t\Delta_t u + AD_t u = 0,$$

on the supposition that D_t and Δ_t are commutative.

Commutate the symbols D_t, Δ_t, and multiply by t^{A-1};

$$\therefore\ t^A\Delta_t . D_t u + At^{A-1}. D_t u = 0.$$

Integrate with Σ_t;

$$\therefore\ t^A D_t u = \Sigma_t 0.$$

Divide by t^A and integrate with S_t;

$$\therefore\ u = S_t t^{-A}\Sigma_t 0 + S_t 0.$$

Ex. 2. To integrate

$$tD_t\Delta_t u + AD_t u + B\Delta_t u = 0,$$

D_t and Δ_t being commutative.

We know that this can be integrated if we can deprive it of one of its last two terms.

Operate on the equation with $f(\Delta_t)$;

$$\therefore\ (tf\Delta_t + f'\Delta_t)\,D_t\Delta_t u + AD_t f(\Delta_t)\,u + B\Delta_t f(\Delta_t)\,u = f(\Delta_t)\,.\,0.$$

Since f is arbitrary we are at liberty to assume

$$f'(\Delta_t)\,D_t\Delta_t + AD_t f(\Delta_t) = 0;$$

$$\therefore\ \Delta_t f'(\Delta_t) + Af(\Delta_t) = 0;$$

$$\therefore\ f\Delta_t = \Delta_t^{-A} = \Sigma_t^{A};$$

$$\text{and}\ \therefore\ tD_t\Delta_t\,.\,f(\Delta_t)\,u + B\Delta_t\,.\,f(\Delta_t)\,u = f(\Delta_t)\,0.$$

$$\therefore\ tD_t\,.\,\Sigma_t^{A-1}u + B\,.\,\Sigma_t^{A-1}u = \Sigma_t^{A}0.$$

Multiply by t^{B-1} and integrate with S_t;

$$\therefore\ t^B\Sigma_t^{A-1}u = S_t t^{B-1}\,\Sigma_t^{A}\,0 + S_t\,0;$$

$$\therefore\ u = \Delta_t^{A-1}t^{-B}S_t t^{B-1}\,\Sigma_t^{A}\,0 + \Delta_t^{A-1}t^{-B}S_t\,0.$$

The form of this integral indicates that unless A be an integer the value of u will be merely symbolical. But no such restriction attaches to B.

If however A be not an integer, and B be integral, we must operate on the proposed equation with $f(D_t)$ instead of $f(\Delta_t)$. The result may be obtained from the above by interchanging A and B, D_t and Δ_t, S_t and Σ_t.

Ex. 3. To integrate

$$(ht + kx)\,D_t\Delta_t u + AD_t u = 0,$$

on the supposition that $D_t x = a$, $\Delta_t x = b$.

The symbols D_t, Δ_t are commutative (Art. 31), and

$$\xi = x - at,\quad \xi' = x - bt.$$

Commutate the symbols and multiply by $(ht+kx)^{m-1}$;

$$\therefore\ (ht+kx)^m\Delta_t\,.\,D_tu+A\,(ht+kx)^{m-1}.\,D_tu=0.$$

Integrate with Σ_t;

$$\therefore\ (ht+kx)^m D_tu=F(\xi'),\ \text{if } A=m\,(h+kb).$$

Divide by $(ht+kx)^m$ and integrate with S_t;

$$\therefore\ u=S_t\,(ht+kx)^{-m}F(\xi')+f(\xi).$$

Ex. 4. To integrate

$$d_t^2u-c^2d_x^2u=\frac{2}{t}\,.\,d_tu.$$

In this example $D_t=d_t+cd_x$, $\Delta_t=d_t-cd_x$, $\xi=x-ct$, $\xi'=x+ct$, and D_t, Δ_t are commutative. Also $2d_t=D_t+\Delta_t$.

$$\therefore\ tD_t\Delta_tu=(D_t+\Delta_t)\,u\ldots\ldots\ldots\ldots\ldots(1).$$

Operate on this equation with $D_t\Delta_t$ (Art. 39, 3);

$$\therefore\ tD_t^2\Delta_t^2u+(D_t+\Delta_t)\,D_t\Delta_tu=(D_t+\Delta_t)\,D_t\Delta_tu;$$

$$\therefore\ D_t^2\Delta_t^2u=0;$$

$$\therefore\ u=S_t^2\,0+\Sigma_t^2\,0$$

$$=tF_1(\xi)+F(\xi)+tf_1(\xi')+f(\xi').$$

The forms of the supernumerary arbitrary functions must be determined by substituting this value of u in equation (1); which being done we find

$$0=F_1(\xi)-cF'(\xi)+f_1(\xi')+cf'(\xi'):$$

and as ξ and ξ' are independent this resolves itself into the two equations

$$F_1(\xi)=cF'(\xi),\ \text{and } f_1(\xi')=-cf'(\xi');$$

$$\therefore\ u=ct\,(F'\xi-f'\xi')+F\xi+f\xi'.$$

Ex. 5. To integrate

$$d_t^2u - c^2d_x^2u = \frac{2u}{t^2}.$$

In this case $D_t = d_t + cd_x$, $\Delta_t = d_t - cd_x$, as in the previous example;

$$\therefore\ D_t\Delta_t u = \frac{2u}{t^2}.$$

Now for u write vt^m, and assume $m(m-1) = 2$, which gives $m = -1$ or 2.

$$\therefore\ tD_t\Delta_t v + m(D_t + \Delta_t)v = 0.$$

If we take the value -1 for m, this equation will coincide with the equation (1) of the last example;

$$\therefore\ ut = v = ct(F'\xi - f'\xi') + F\xi + f\xi';$$

$$\therefore\ u = c(F'\xi - f'\xi') + t^{-1}(F\xi + f\xi').$$

Ex. 6. To integrate

$$d_t^2u - c^2d_x^2u = \frac{Au}{(ht+kx)^2}.$$

In this example D_t, Δ_t retain the same values as in the last example. Write $v(ht+kx)^m$ for u, and assume

$$m(m-1)(h^2 - k^2c^2) = A.$$

$$\therefore\ (ht+kx)D_t\Delta_t v + m(h-kc)D_t v + m(h+kc)\Delta_t v = 0.$$

Operate on this with $f(D_t\Delta_t)$; (for brevity we may write f for this function)

$$\therefore\ \{(ht+kx)f + (h-kc)D_t.f' + (h+kc)\Delta_t.f'\}D_t\Delta_t v$$
$$+ m(h-kc)D_tf.v + m(h+kc)\Delta_tf.v = f(D_t\Delta_t)\,0.$$

Now as f is arbitrary we are at liberty to assume

$$D_t . f'(D_t\Delta_t) . D_t\Delta_t + mD_t f(D_t\Delta_t) = 0 ;$$

$$\therefore \; D_t\Delta_t . f'(D_t\Delta_t) + mf(D_t\Delta_t) = 0 ;$$

$$\therefore \; f(D_t\Delta_t) = (D_t\Delta_t)^{-m} = (S_t\Sigma_t)^m .$$

$$\therefore \; (ht + kx)(S_t\Sigma_t)^{m-1} v = (S_t\Sigma_t)^m 0 = S_t^m 0 + \Sigma_t^m 0 ;$$

$$\therefore \; v = D_t^{m-1}\Delta_t^{m-1}(ht + kx)^{-1}(S_t^m 0 + \Sigma_t^m 0).$$

This formula indicates that m must be an integer; and that the value of v consists of two independent parts, one depending on $S_t^m 0$, and the other on $\Sigma_t^m 0$. We may find them separately and similarly.

Now

$$S_t^m 0 = F_0(\xi) + F_1(\xi) . (ht + kx) + \ldots + F_{m-1}(\xi) . (ht + kx)^{m-1},$$

$$(ht + kx)^{-1} S_t^m 0 = (ht + kx)^{-1} F_0(\xi) + F_1(\xi) + \ldots$$
$$\ldots + (ht + kx)^{m-2} F_{m-1}(\xi) ;$$

$$\therefore \; D_t^{m-1}(ht + kx)^{-1} S_t^m 0 = (ht + kx)^{-m} F(\xi).$$

Similarly

$$\Delta_t^{m-1}(ht + kx)^{-1}\Sigma_t^m 0 = (ht + kx)^{-m} f(\xi') ;$$

$$\therefore \; v = \Delta_t^{m-1}(ht + kx)^{-m} F(\xi) + D_t^{m-1}(ht + kx)^{-m} f(\xi') ;$$

u will be known from the equation

$$u = v(ht + kx)^m .$$

Ex. 7. To integrate

$$d_t^2 u - c^2 d_x^2 u = \frac{2A}{t} . d_t u .$$

This differs from Example (4) in the last term only, having $2A$ for 2;

$$\therefore \; tD_t\Delta_t u = A(D_t + \Delta_t) u \ldots\ldots\ldots\ldots (1).$$

Operate on this equation with $f(D_t\Delta_t)$,

$$\therefore\ \{tf(D_t\Delta_t) + (D_t + \Delta_t) f'(D_t\Delta_t)\} D_t\Delta_t u$$
$$= A\,(D_t + \Delta_t) f(D_t\Delta_t)\, u + f(D_t\Delta_t)\, 0.$$

We are at liberty to assume

$$D_t\Delta_t f'(D_t\Delta_t) = Af(D_t\Delta_t);$$
$$\therefore f(D_t\Delta_t) = (D_t\Delta_t)^A,$$

and $$t\,(D_t\Delta_t)^{A+1} u = (D_t\Delta_t)^A\, 0 = 0;$$
$$\therefore\ u = (S_t\Delta_t)^{A+1}\, 0 = S_t^{A+1}\, 0 + \Sigma_t^{A+1}\, 0.$$

We may find the two parts of u separately and similarly.

Now

$$S_t^{A+1}\, 0 = F_1(\xi) + tF_2(\xi) + t^2F_3(\xi) + \ldots + t^AF_{A+1}(\xi).$$

The forms of the supernumerary arbitrary functions must be determined by substituting this expression for u in equation (1). We thus find that

$$F_{n+2}(\xi) = \frac{2c}{n+1}\cdot\frac{A-n}{2A-n}\cdot F'_{n+1}(\xi).$$

In a similar way we find

$$f_{n+2}(\xi) = \frac{-2c}{n+1}\cdot\frac{A-n}{2A-n}\cdot f'_{n+1}(\xi');$$

$$\therefore\ u = (F\xi + f\xi') + ct\,(d_\xi F\xi - d_{\xi'} f\xi')$$
$$+ \frac{A-1}{2A-1}\cdot\frac{(2ct)^2}{1\,.\,2}\cdot(d_\xi^2 F\xi + d_{\xi'}^2 f\xi')$$
$$+ \frac{A-1}{2A-1}\cdot\frac{A-2}{2A-2}\cdot\frac{(2ct)^3}{1\,.\,2\,.\,3}\cdot(d_\xi^3 F\xi - d_{\xi'}^3 f\xi') + \ldots\ldots$$

Remark. If A be negative, write $-A$ for A, and the integral will then be

$$u = \Delta_t^{A-1} . t^{-A} F(\xi) + D_t^{A-1} . t^{-A} f(\xi').$$

Ex. 8. To integrate

$$d_t^2 u - d_x^2 u + \frac{4 d_t u}{t+x} = 0.$$

Here $D_t = d_t + d_x$, $\Delta_t = d_t - d_x$, $\xi = x - t$, $\xi' = x + t$, $2d_t = D_t + \Delta_t$, and D_t, Δ_t are commutative. The equation becomes

$$(t+x) D_t \Delta_t u + 2\Delta_t u + 2 D_t u = 0.$$

Now since $t + x = \xi'$, therefore $D_t \xi' = 2$; and we may integrate at once with S_t;

$$\therefore \xi' \Delta_t u + 2u = F(\xi).$$

As ξ' is in this equation a quasi-constant, multiply by $\epsilon^{\frac{2t}{\xi'}}$ and integrate with Σ_t;

$$\therefore u \epsilon^{\frac{2t}{\xi'}} = \Sigma_t \epsilon^{\frac{2t}{\xi'}} F(\xi) + f(\xi').$$

Ex. 9. To integrate

$$d_t (d_t^2 u - d_x^2 u) + d_x (d_t^2 u - d_x^2 u) = \frac{2}{t+x} . (d_t^2 u - d_x^2 u).$$

Here D_t and Δ_t have the same values as in the last Example.

$$\therefore (d_t + d_x) D_t \Delta_t u - \frac{2}{t+x} . D_t \Delta_t u = 0;$$

$$\therefore D_t . D_t \Delta_t u - \frac{D_t (t+x)}{t+x} . D_t \Delta_t u = 0.$$

Divide by $t+x$, and integrate with S_t,

$$\therefore\ D_t\Delta_t u = (t+x)\,F'(\xi) = \xi' F'(\xi).$$

Now as ξ' is a quasi-constant in reference to Σ_t, we have

$$\begin{aligned} u &= S_t\xi' \,.\, \Sigma_t F'(\xi) + S_t\,0 + \Sigma_t\,0 \\ &= \frac{1}{4}\xi'^2 \,.\, F(\xi) + \phi\,(\xi) + f(\xi'), \\ &= \frac{1}{4}(t+x)^2 F\,(t-x) + \phi\,(t-x) + f\,(t+x). \end{aligned}$$

REMARK. $\Sigma_t F'(\xi)$ is not exactly equal to $F(\xi)$, but to $-\frac{1}{2}F(\xi)$, but the constant $\left(-\frac{1}{2}\right)$ may be considered as absorbed into the arbitrary function.

The form of the above integral may be simplified by writing $(t-x)^2+4tx$ instead of $(t+x)^2$ in the first term. It may then be written in the form

$$u = txF\,(t-x) + \phi\,(t-x) + f\,(t+x).$$

Ex. 10. To integrate

$$d_t^2 u + (a+b)\,d_t d_x u + ab d_x^2 u = \frac{Cu}{(ht+kx)^2}.$$

Here

$$D_t = d_t + ad_x,\quad \Delta_t = d_t + bd_x,\quad \xi = x - at,\quad \xi' = x - bt;$$

$$\therefore\ (ht+kx)^2 D_t\Delta_t u = Cu.$$

If we write $(ht+kx)^m v$ for u we obtain, assuming

$$m\,(m-1)\,(h+ka)\,(h+kb) = C,$$

$$(ht+kx)\,D_t\Delta_t v + m\,(h+kb)\,D_t v + m\,(h+ka)\,\Delta_t v = 0.$$

This equation has already been treated in Example (6).

Ex. 11. To integrate

$$d_t^2u + 2ad_td_xu + a^2d_x^2u = \frac{Cu}{(ht+kx)^2}.$$

Here $D_t = d_t + ad_x$, $\xi = x - at$.

$$\therefore\ (ht+kx)^2 . D_t^2u = Cu.$$

The integral of which is

$$u = (ht+kx)^m F(\xi) + (ht+kx)^n f(\xi),$$

m and n being the roots of

$$m(m-1)(h+ka)^2 = C.$$

C may be a quasi-constant.

Ex. 12. To integrate $d_t^2u - \frac{c^2}{t^4} . d_x^2u = 0$.

This falls under the general form considered in Art. 50, Ex. 3. We here assume $t^{-1} = \tau$, and change the independent from t to τ.

$$\therefore\ \tau(d_\tau^2u - c^2d_x^2u) + 2d_\tau u = 0.$$

[The rule for determining in equations of this form what power of t is to be assumed for τ is this. The power of t in the question is t^{-4}, take the square root of this $(=t^{-2})$ and integrate this $(=t^{-1})$. This is the value of τ. (If the form of t in the question had been t^{-2}, its square root would be t^{-1}, and $\therefore\ \tau = \log t$.)]

$$\therefore\ D_\tau = d_\tau + cd_x,\quad \Delta_\tau = d_\tau - cd_x,\quad \xi = x - c\tau,\quad \xi' = x + c\tau;$$

$$\therefore\ \tau D_\tau\Delta_\tau u + (D_\tau + \Delta_\tau)u = 0.$$

Operate with $(D_\tau\Delta_\tau)^{-1}$,

$$\therefore\ \tau D_\tau\Delta_\tau(D_\tau\Delta_\tau)^{-1}u = (D_t\Delta_t)^{-1}0 = S_t\Sigma_t 0;$$

$$\therefore\ \tau u = S_t 0 + \Sigma_t 0\,;$$

$$\therefore\ \frac{u}{t} = F(\xi) + f(\xi')$$

$$= F(x - c\tau) + f(x + c\tau)\,;$$

$$\therefore\ u = tF\left(x - \frac{c}{t}\right) + tf\left(x + \frac{c}{t}\right).$$

Ex. 13. To integrate $d_t^2 u - c^2 t^{-\frac{4}{3}} d_x^2 u = 0$.

In this case we must assume $t^{\frac{1}{3}} = \tau$.

$$\therefore\ \tau\,(d_\tau^2 u - c^2 d_x^2 u) = 2d_\tau u\,;$$

$$\therefore\ \tau D_\tau \Delta_\tau u = (D_\tau + \Delta_\tau)\,u \ldots\ldots\ldots\ldots\ldots (1).$$

Operate on this equation with $(D_\tau \Delta_\tau)$,

$$\therefore\ \tau\,(D_\tau \Delta_\tau)^2 u = 0\,;$$

$$\therefore\ u = S_\tau^2 0 + \Sigma_\tau^2 0$$

$$= F(\xi) + \tau F_1(\xi) + f(\xi') + \tau f_1(\xi').$$

Substituting $F(\xi) + \tau F_1(\xi)$ in equation (1) for u we find $F_1(\xi) = cF'(\xi)$. Similarly $f_1(\xi') = -cf'(\xi')$.

$$\therefore\ u = c\tau\,(F'\xi - f'\xi) + F\xi + f\xi'$$

$$= ct^{\frac{1}{3}}\,\{F'(x - ct^{\frac{1}{3}}) - f'(x + ct^{\frac{1}{3}})\}$$

$$+ F(x - ct^{\frac{1}{3}}) + f(x + ct^{\frac{1}{3}}).$$

Ex. 14. To integrate $D_t^2 u - c^2 t^{-4} u = 0$.

The integral of $d_t^2 u - c^2 t^{-4} u = 0$ is

$$\frac{u}{t} = C_1 \epsilon^{\frac{c}{t}} + C_2 \epsilon^{-\frac{c}{t}}\,;$$

consequently the integral of the proposed equation is

$$\frac{u}{t} = \epsilon^{\frac{c}{t}} F(\xi, \eta, \zeta\ldots) + \epsilon^{-\frac{c}{t}} f(\xi, \eta, \zeta\ldots),$$

and c may be a quasi-constant, or a function of d_x, d_y, ...

Ex. 15. To integrate

$$d_t^2 u + \frac{2x}{t} \,.\, d_t d_x u + \frac{x^2}{t^2} \,.\, d_x^2 u - c^2 x^{-4} u = 0.$$

Here $D_t = d_t + \frac{x}{t} \,.\, d_x$, $\xi = \frac{x}{t}$, and $x^{-4} = \frac{t^4}{x^4} \,.\, t^{-4} = \xi^{-4} \,.\, t^{-4}$;

$$\therefore\ D_t^2 u - c^2 \xi^{-4} \,.\, t^{-4} u = 0.$$

This agrees with the form of the preceding Example, with $c\xi^{-2}$ for c.

$$\therefore\ \frac{u}{t} = \epsilon^{\frac{c}{t\xi^2}} F(\xi) + \epsilon^{-\frac{c}{t\xi^2}} f(\xi)$$

$$= \epsilon^{\frac{ct}{x^2}} F\left(\frac{x}{t}\right) + \epsilon^{-\frac{ct}{x^2}} f\left(\frac{x}{t}\right).$$

Ex. 16. To integrate $d_t^2 u + cu = \frac{A(A-1)}{t^2} \,.\, u$; A being an integer.

Multiply by t^2;

$$\therefore\ (t^2 d_t^2 + ct^2)\, u = A(A-1)\, u.$$

For $t^2 d_t^2$ write $d_\theta (d_\theta - 1)$, (Art. 42, 5);

$$\therefore\ \{d_\theta (d_\theta - 1) - A(A-1)\} \,.\, u + ct^2 u = 0;$$

$$\therefore\ (d_\theta - A)(d_\theta + A - 1)\, u + ct^2 u = 0.$$

Diminish the symbolic factorials by A (Art. 45);

$$\therefore\ (d_\theta - 2A)(d_\theta - 1) \,.\, ut^A + ct^2 \,.\, ut^A = 0,$$

or, writing v for ut^A,

$$(d_\theta - 2A)(d_\theta - 1)v + ct^2v = 0.$$

We have now to increase the first of the factorials by nk (that is, by $2A$, for $k = 2$, and $n = -A$) to reduce them to the requisite form (Art. 44). We must therefore assume (Art. 49)

$$(t^{-2}d_\theta)^{-A}.vt^{-2A} = (t^{-2})^{-A}.wt^{-2A} = w;$$

$$\therefore d_\theta(d_\theta - 1)w + ct^2w = 0;$$

$$\therefore t^2d_t^2w + ct^2w = 0;$$

$$\therefore d_t^2w + cw = 0.$$

The integral of this equation is known: and u will be known from the equations

$$u = vt^{-A}, \text{ and } v = t^{2A}(t^{-2}d_\theta)^A w = t^{2A}(t^{-1}d_t)^A w.$$

$$\therefore u = t^A(t^{-1}d_t)^A w.$$

For d_t in this example we may write D_t, and c may be a quasi-constant: if d_t be used, then c may be a function of $d_x, d_y, \ldots$.

Ex. 17. To integrate

$$d_t^2u + Ad_tu = \frac{2u}{t^2}.$$

This may be written,

$$d_t(d_t + A)u = \frac{2u}{t^2}.$$

For u write $vt^{-\frac{1}{2}A}$;

$$\therefore \left(d_t - \frac{1}{2}A\right)\left(d_t + \frac{1}{2}A\right)v = \frac{2v}{t^2};$$

$$\therefore\ d_t^2 v - \frac{1}{4} A^2 v = \frac{2v}{t^2},$$

which falls under the last example. We may write D_t for d_t, and a quasi-constant for A.

Ex. 18. To integrate

$$d_t^2 u + a d_t d_x u = \frac{2u}{t^2}.$$

This is equivalent to,

$$d_t (d_t + a d_x)\, u = \frac{2u}{t^2};$$

$$\therefore\ D_t = d_t + a d_x, \text{ and } \xi = x - at;$$

$$\therefore\ d_t D_t u = \frac{2u}{t^2}.$$

Now assume $u = vt^m$, and put $m(m-1) = 2$, which gives $m = 2$, or -1.

Taking the latter, we have

$$t d_t D_t v = (D_t + d_t)\, v \quad \text{..................} (1).$$

Differentiate with $d_t D_t$,

$$\therefore\ d_t^2 D_t^2 v = 0;$$

$$\therefore\ v = s_t^2\, 0 + S_t^2\, 0$$

$$= tF_1(x) + F(x) + tf_1(\xi) + f(\xi).$$

The supernumerary constants are to be determined by substituting this result in equation (1), (see Ex. 4);

$$\therefore\ v = \frac{at}{2}(f'\xi - F'x) + f\xi + Fx;$$

$$\therefore\ u = \frac{a}{2}(f'\xi - F'x) + \frac{1}{t}(f\xi + Fx).$$

Ex. 19. To integrate

$$t^2 d_t^2 u + 2tx d_t d_x u + x^2 d_x^2 u = t^m x^n.$$

Dividing by t^2,

$$d_t^2 u + \frac{2x}{t} d_t d_x u + \frac{x^2}{t^2} d_x^2 u = t^{m-2} x^n ;$$

$$\therefore\ D_t = d_t + \frac{x}{t} d_x, \text{ and } \xi = \frac{x}{t} ;$$

$$\therefore\ D_t^2 u = t^{m-2} . (\xi t)^n = \xi^n . t^{m+n-2} ;$$

$$\therefore\ u = \xi^n \frac{t^{m+n}}{(m+n-1)(m+n)} + tF(\xi) + f(\xi)$$

$$= \frac{t^m x^n}{(m+n-1)(m+n)} + tF\left(\frac{x}{t}\right) + f\left(\frac{x}{t}\right).$$

Ex. 20. To integrate

$$t^n d_t^n u + n t^{n-1} x d_t^{n-1} d_x u + n \frac{n-1}{2} t^{n-2} x^2 d_t^{n-2} d_x^2 u + \dots$$
$$+ x^n d_x^n u = t^m \phi\left(\frac{x}{t}\right).$$

Divide by t^n, then

$$D_t = d_t + \frac{x}{t} d_x, \text{ and } \xi = \frac{x}{t} ;$$

$$\therefore\ D_t^n u = t^{m-n} \phi(\xi) ;$$

$$\therefore\ u = \frac{t^m \phi(\xi)}{m(m-1)(m-2)\dots(m-n+1)}$$
$$+ F_1(\xi) + tF_2(\xi) + t^2 F_3(\xi) + \dots + t^{n-1} F_n(\xi).$$

Ex. 21. To integrate

$$t^2 d_t^2 u - x^2 d_x^2 u + t d_t u - x d_x u = 0.$$

$$\therefore\ d_t^2 u - \frac{x^2}{t^2} . d_x^2 u + \frac{1}{t}\left(d_t u - \frac{x}{t} . d_x u\right) = 0.$$

Comparing this with the formula (1) Art. 30,

$$\Delta_t D_t u = d_t^2 u + (D_t x + \Delta_t x)\, d_t d_x u + D_t x \,.\, \Delta_t x \,.\, d_x^2 u + \Delta_t D_t x \,.\, d_x u,$$

we have $D_t x = \frac{x}{t}$, $\Delta_t x = -\frac{x}{t}$; $\xi = \frac{x}{t}$, $\xi' = tx$,

$$\therefore\ \Delta_t D_t x = -\frac{2x}{t^2}, \text{ and } D_t \Delta_t x = 0,$$

hence D_t and Δ_t are not commutative;

$$\therefore\ t\Delta_t D_t u + D_t u = 0.$$

Integrate with Σ_t,

$$\therefore\ tD_t u = F_1(\xi') = F_1(\xi t^2) = 2\xi t^2 F'(\xi t^2);$$

$$\therefore\ D_t u = 2\xi t F'(\xi t^2);$$

$$\therefore\ u = F(\xi t^2) + f(\xi)$$

$$= F(tx) + f\left(\frac{x}{t}\right).$$

Ex. 22. To integrate

$$d_t^2 u + \frac{2}{t} \,.\, d_t u - a^2 d_x^2 u = \frac{2}{t^2} \,.\, u.$$

$$\therefore\ t^2 d_t^2 u + 2t d_t u - a^2 t^2 d_x^2 u = 2u;$$

$$\therefore\ \{d_\theta (d_\theta - 1) + 2d_\theta - 2\}\, u - a^2 t^2 d_x^2 u = 0;$$

$$\therefore\ (d_\theta + 2)(d_\theta - 1)\, u - a^2 t^2 d_x^2 u = 0.$$

For convenience write C for $-a^2 d_x^2$;

$$\therefore\ (d_\theta + 2)(d_\theta - 1)\, u + Ct^2 u = 0.$$

We have now to diminish the factor $d_\theta + 2$ by 2; this we do by assuming (see Art. 49)

$$(t^{-2} d_\theta) \,.\, ut^2 = (t^{-2}) \,.\, vt^2 = v \quad \ldots\ldots\ldots\ldots (1);$$

$$\therefore\ d_\theta (d_\theta - 1)\, v + Ct^2 v = 0\,;$$

$$\therefore\ d_t^2 v + Cv = 0\,;$$

$$\therefore\ d_t^2 v - a^2 d_x^{\,2} v = 0\,;$$

$$\therefore\ v = F(x - at) + f(x + at).$$

From equation (1), we have

$$u = t^{-2} s_t\,(tv).$$

Ex. 23. To integrate

$$d_t d_x u + \frac{d_t u + d_x u}{t + x} = \frac{2u}{(t + x)^2}\,.$$

For u write $v\,(t + x)^m$, and put $m\,(m + 1) = 2$ which gives $m = 1$ or -2. We take the latter value of m;

$$\therefore\ (t + x)\, d_t d_x v = d_t v + d_x v \ \ldots\ldots\ldots\ldots\ldots (1).$$

Differentiate this with $d_t d_x$, (Art. 39, 2);

$$\therefore\ (t + x)\, d_t^2 d_x^{\,2} v = 0\,;$$

$$\therefore\ v = s_t^2 s_x^{\,2}\, 0 = s_t^2\, 0 + s_x^{\,2}\, 0$$

$$= (t + x)\, F_1(x) + F(x) + (t + x)\, f_1(t) + f(t).$$

The forms of the supernumerary arbitrary functions are to be determined by substituting this value of v in equation (1). This gives

$$F_1(x) = -\frac{1}{2} F'(x), \text{ and } f_1(t) = -\frac{1}{2} f'(t).$$

$$\therefore\ u = v\,(t + x)^{-2} = (t + x)^{-2}\,(Fx + ft) - \frac{1}{2}\,(t + x)^{-1}\,(F'x + f't).$$

Ex. 24. To integrate

$$(t - x)\,(d_t^2 u - d_x^{\,2} u) = 4.$$

In this example

$$D_t = d_t + d_x,\ \Delta_t = d_t - d_x,\ \xi = x - t,\ \xi' = x + t;$$

$$\therefore\ (t - x)\, D_t\Delta_t u = 4;$$

$$\therefore\ D_t\Delta_t u = -\frac{4}{\xi} = \frac{2\Delta_t\xi}{\xi},\ (\Delta_t\xi = -2).$$

Integrate with Σ_t,

$$\therefore\ D_t u = 2\log\xi + F'(\xi');$$

$$\therefore\ u = 2t\log\xi + F(\xi') + f(\xi).$$

$S_tF'(\xi')$ is not equal to $F(\xi')$; but it is equivalent to it in this case, because $D_t\xi'\,(=2)$ is constant, and the constant may be considered as being absorbed in the arbitrary function.

CHAPTER X.

EQUATIONS OF THE SECOND ORDER. COEFFICIENTS FUNCTIONS OF p AND q.

54. IN this Chapter it will be a convenience to represent $d_t u$ and $d_x u$ by p and q respectively. The following are the formulæ suitable for this class of equations. They are a mere adaptation of the general formula of Art. 7 to these equations, and no new principle is involved in them.

$$D_t p = d_t p + D_t x \,.\, d_x p = d_t^2 u + D_t x \,.\, d_t d_x u \,\ldots\ldots\, (1),$$

$$D_t q = d_t q + D_t x \,.\, d_x q = d_t d_x u + D_t x \,.\, d_x^2 u \,\ldots\ldots\, (2).$$

From these we immediately deduce the following forms, which it will be observed are such as can be at once adapted to those terms of an equation which are of the second order. They are the forms under which such terms generally present themselves, and will serve as guides in choosing the starting-point in commencing the integration of any proposed equation.

$$D_t p - D_t x \,.\, D_t q = d_t^2 u - (D_t x)^2 \,.\, d_t^2 u \,\ldots\ldots\ldots\ldots\ldots\, (3),$$

$$D_t p + D_t x \,.\, D_t q = d_t^2 u + 2 D_t x \,.\, d_t d_x u + (D_t x)^2 \,.\, d_t^2 u \,\ldots\ldots\, (4),$$

$$D_t p + M \,.\, D_t q = d_t^2 u + (D_t x + M)\, d_t d_x u + M D_t x \,.\, d_x^2 u \ldots (5).$$

In the last of these forms M is arbitrary. In using the form (4) the value of D_tx will be definitely known by comparison of the right-hand side of the formula with the proposed equation; and it will have only one value; in consequence of which there will be only *one* first integral of the equations that fall under that form. The general integral corresponding thereto may contain two arbitrary functions, but they will be functions of the *same* subject.

In using forms (3) and (5) the arbitrary functions, which will appear in the general integral of equations which fall under those forms, will have *different* subjects. For in form (3) the value of $(D_tx)^2$ will be given by comparison of the right-hand side with the proposed equation; and consequently D_tx may be either (+) or (−) a certain quantity. Hence there may be *two* first integrals, one of them to be found by using the (+) value, and the other by using the (−) value of D_tx. Also in using form (5) the quantities that will be known by comparison of the right-hand side with a proposed equation are $D_tx + M$ and $M . D_tx$, in which M and D_tx are evidently interchangeable; consequently by using first one value for D_tx and the other for M, and then interchanging them, we have two different values for D_tx, and consequently there may be *two* first integrals, and two *different* subjects of the arbitrary functions that enter into the final integral.

Examples suitable for this Chapter are not very abundant in Treatises on Partial Differential Equations, but of those that are usually given there are some which are interesting from their connection with Geometry or Physics. These introduce us to a new class of integration difficulties,—and in consequence of these, explanations are required which it is by no means easy to give; and of those explanations that are here given conjecture forms a large feature, but it has been judged better to offer conjectural explanations than to pass over the difficulties in silence. Some good will have been effected if nothing more is done here than to direct the attention of mathematicians to them. The difficulties here referred to are brought to light by the method of integration peculiar to this Essay.

Ex. 1. To integrate

$$q^2 d_t^2 u - 2pq d_t d_x u + p^2 d_x^2 u = 0.$$

Divide by q^2,

$$\therefore\ d_t^2 u - \frac{2p}{q} d_t d_x u + \frac{p^2}{q^2} d_x^2 u = 0.$$

On comparing this with formula (4), we find that

$$D_t x = -\frac{p}{q}, \text{ and } D_t p + D_t x \,.\, D_t q = 0;$$

$$\therefore\ p + D_t x \,.\, q = 0, \text{ and } D_t p - \frac{p}{q} \,.\, D_t q = 0;$$

$$\therefore\ D_t u = 0, \text{ and } \frac{qD_t p - pD_t q}{q^2} = 0;$$

$$\therefore\ u = \xi, \text{ and } \frac{p}{q} = F(\xi);$$

$$\therefore\ -D_t x = F(\xi);$$

$$\therefore\ -x = tF(\xi) + f(\xi),$$

$$\therefore\ 0 = x + tF(u) + f(u).$$

This is the general equation of a certain class of *Ruled Surfaces.*

Ex. 2. To integrate

$$d_t^2 u \,.\, d_x^2 u - (d_t d_x u)^2 = 0.$$

This is the characteristic equation of developable surfaces. Its terms are not linear, but from its peculiar form it can be integrated by means of the above formulæ.

Divide by d_x^2u,

$$\therefore\ d_t^2u - \frac{d_td_xu}{d_x^2u}\, d_td_xu = 0.$$

Comparing this with form (1) we have

$$D_tp = 0, \text{ and } D_tx = -\frac{d_td_xu}{d_x^2u};$$

$$\therefore\ p = \xi, \text{ and } d_td_xu + D_tx \,.\, d_x^2u = 0.$$

And by form (2) we have

$$D_tq = d_td_xu + D_tx \,.\, d_x^2u = 0;$$

$$\therefore\ q = F(\xi) = F(p).$$

This is the first integral. The final integral may be thus found:

$$D_tu = p + D_tx \,.\, q = \xi + D_tx \,.\, F(\xi);$$

$$\therefore\ u = \xi t + xF(\xi) + f(\xi).$$

But this is subject to the condition that $\xi = p = d_tu$;

$$\therefore\ \xi = \xi + \{t + xF'(\xi) + f'(\xi)\} \,.\, d_t\xi;$$

$$\therefore \text{ either } 0 = t + xF'(\xi) + f'(\xi), \text{ or } 0 = d_t\xi = d_t^2u.$$

Now in *general* d_t^2u cannot be $=0$, therefore the general integral of the proposed equation is that which results from the elimination of ξ between the two equations

$$\begin{cases} u = \xi t + xF\xi + f\xi, \\ 0 = t + xF'\xi + f'\xi. \end{cases}$$

55. Let us inquire to what kind of developable surfaces the condition $d_t\xi = 0$ belongs.

Since $d_t\xi = d_t^2u$, this being substituted in the original equation gives $d_td_xu = 0$. We have therefore to find that form of u which will satisfy simultaneously the two equations

$$d_t^2u = 0, \text{ and } d_xd_tu = 0;$$

$$\therefore\ u = Ct + \phi(x),$$

C being an arbitrary constant and ϕ an arbitrary function. This is the equation of a cylindrical surface. And therefore when the developable surface degenerates into a cylinder, the integral just found must take the place of the general integral.

We may remark that the proposed equation when written in the form $\frac{d_tp}{d_xp} = \frac{d_tq}{d_xq}$ falls under Ex. 6, Chap. 2; and consequently we have at once, as its first integral, $q = F(p)$.

Ex. 3. To integrate

$$qd_t^2u - pd_td_xu = 0.$$

This equation, like the preceding one, may be put under the form of Chap. 2, Ex. 6, for it may be written in the form

$$\frac{d_tp}{d_xp} = \frac{d_tu}{d_xu};$$

$$\therefore\ p = F(u) = \frac{1}{f'(u)} \text{ (suppose)};$$

$$\therefore\ f'(u).d_tu = 1;$$

$$\therefore\ fu = t + \phi(x).$$

Ex. 4. To integrate

$$q(1+q)\,d_t^2u-(p+q+2pq)\,d_td_xu+p(1+p)\,d_x^2u=0.$$

Divide by $q(1+q)$,

$$\therefore\ d_t^2u-\left(\frac{p}{q}+\frac{1+p}{1+q}\right)d_td_xu+\frac{p}{q}\cdot\frac{1+p}{1+q}\,d_x^2u=0.$$

On comparing this with form (5) of this Chapter, we have the option of assuming

$$D_tx=-\frac{p}{q},\ \text{or}\ =-\frac{1+p}{1+q};$$

the former on trial is found to be the simpler. We therefore assume

$$M=-\frac{1+p}{1+q},\quad D_tx=-\frac{p}{q},\quad\text{and}\quad D_tp-\frac{1+p}{1+q}\cdot D_tq=0;$$

$$\therefore\ D_tu=p+D_tx\,.\,q=0,\quad\text{and}\quad\frac{D_tp}{1+p}-\frac{D_tq}{1+q}=0;$$

$$\therefore\ u=\xi,\ \text{and}\ 1+p=(1+q)\,F(\xi);$$

$$\therefore\ p-F(\xi)\,q=F(\xi)-1,$$

$$\text{or}\ p-F(u)\,q=F(u)-1.$$

This is an equation of the first order, and may be integrated in the usual manner.

Compare it with the fundamental equation

$$\Delta_tu=p+\Delta_tx\,.\,q;$$

$$\therefore\ \Delta_tu=F(u)-1,\ \text{and}\ \Delta_tx=-F(u).$$

$$\therefore\ \Delta_t u + \Delta_t x + 1 = 0, \text{ and } \frac{\Delta_t u}{F(u) - 1} = 1\,;$$

$$\therefore\ u + x + t = \xi', \text{ and } \phi u = t + f(\xi')\,;$$

$$\therefore\ \phi(u) = t + f(u + x + t)$$

is the integral required.

Ex. 5. To integrate

$$(1 + pq + q^2)\, d_t^2 u + (q^2 - p^2)\, d_t d_x u - (1 + pq + p^2)\, d_x^2 u = 0.$$

We notice that

$$q^2 - p^2 = (1 + pq + q^2) - (1 + pq + p^2)\,;$$

let this be substituted.

$$\therefore\ d_t(p + q) - \frac{1 + pq + p^2}{1 + pq + q^2} \,.\, d_x(p + q) = 0.$$

Compare this with the general formula

$$D_t(p + q) = d_t(p + q) + D_t x \,.\, d_x(p + q)\,;$$

$$\therefore\ D_t(p + q) = 0, \text{ and } D_t x = -\frac{1 + pq + p^2}{1 + pq + q^2}\,;$$

$$\therefore\ p + q = \xi, \text{ and } D_t x = -\frac{1 + p\xi}{1 + q\xi}\,;$$

$$\therefore\ 1 + D_t x + \xi(p + D_t x \,.\, q) = 0.$$

Integrate with S_t, remembering that $p + D_t x \,.\, q = D_t u$,

$$\therefore\ t + x + \xi u = F(\xi) \ldots\ldots\ldots\ldots\ldots (1)\,;$$

$$\therefore\ t + x + (p + q)\, u = F(p + q).$$

This is a first integral of the proposed equation.

This equation may be integrated in the following manner.

By inspection of equation (1), we perceive that

$$\xi = p + q = \phi\,(u,\ t + x).$$

Comparing this with the general formula,

$$\Delta_t u = p + \Delta_t x \,.\, q,$$

we find

$$\Delta_t x = 1, \text{ and } \Delta_t u = \phi\,(u,\ t + x) = \xi\,;$$

$$\therefore\ x = t + \xi', \text{ and } t + x + u\xi = F\,(\xi) \text{ from (1).}$$

Divide the last equation by ξ, and differentiate with Δ_t,

$$\therefore\ \frac{2 + \xi^2}{\xi} - (t + x)\,.\,\frac{\Delta_t\xi}{\xi^2} = \Delta_t\,.\,\frac{F(\xi)}{\xi}\,.$$

Multiply by 2, which is $= \Delta_t\,(t + x)$, and divide by $\sqrt{2 + \xi^2}$;

$$\therefore\ \frac{\sqrt{2 + \xi^2}}{\xi}\,.\,\Delta_t\,(t + x) - (t + x)\,.\,\frac{2\Delta_t\xi}{\xi^2\sqrt{2 + \xi^2}} = \frac{2}{\sqrt{2 + \xi^2}}\,.\,\Delta_t\,\frac{F\xi}{\xi}\,.$$

Integrate with Σ_t;

$$\therefore\ \frac{\sqrt{2 + \xi^2}}{\xi}\,.\,(t + x) = \Sigma_t\,\frac{2\Delta_t\,(\xi^{-1}F\xi)}{\sqrt{2 + \xi^2}} + f(\xi')\ \ldots\ldots(2).$$

If now we eliminate ξ between this equation and equation (1), and write $x - t$ for ξ', we shall have the required integral.

But equation (2) contains a term affected with the sign of integration. We are at liberty to write $F_1\,(\xi)$ for that term, but this would introduce the sign of integration into equation (1).

The integral of the proposed equation may be found in a more convenient shape from the second first integral, as follows:

The proposed equation may be thrown into the form,

$$(1+pq+q^2)(d_t^2u+d_td_xu)=(1+pq+p^2)(d_td_xu+d_x^2u).$$

On reference to formulæ (1) and (2), we see that if we assume $\Delta_t x=1$, which gives $\xi'=x-t$, then

$$(1+pq+q^2)\,\Delta_t p=(1+pq+p^2)\,\Delta_t q.$$

For brevity write ξ for $p+q$, and express this equation in terms of $p+q$ and $p-q$;

$$\therefore\ (p-q)\,\xi\Delta_t\xi=(2+\xi^2)\,\Delta_t(p-q).$$

Integrate with Σ_t,

$$\therefore\ \sqrt{2+\xi^2}=(p-q)\,.f(\xi')\ \dots\dots\dots\dots\dots(3).$$

This is the second first integral. It may be integrated in the following way:

$$\begin{aligned}2du&=2pdt+2qdx\\&=(p+q)(dt+dx)+(p-q)(dt-dx)\\&=\xi(dt+dx)-\sqrt{2+\xi^2}\,.\frac{d\xi'}{f(\xi')}\ \text{ from (3)};\end{aligned}$$

$$\left.\begin{aligned}\therefore\ 2u&=\xi(t+x)-\sqrt{2+\xi^2}\,.f_1(\xi')+F_1(\xi)\\0&=t+x-\frac{\xi f_1(\xi')}{\sqrt{2+\xi^2}}+F_1'(\xi)\end{aligned}\right\}\dots\dots\dots\dots(4).$$

If we eliminate ξ between these equations, and write $x-t$ for ξ', we shall have the required integral.

The advantage of this integral over the one before found in (1) and (2) is, that it contains no quantities under the sign of integration.

The above Example may be put under a somewhat more general form which can be integrated in the same manner. It is here added as an exercise for the student.

To integrate

$$(k + pq + aq^2)(d_t^2 u + a d_t d_x u) = (h + apq + p^2)(d_t d_x u + a d_x^2 u).$$

Ex. 6. To integrate

$$q^2 d_t^2 u - p^2 d_x^2 u = 0.$$

Divide by q^2 and compare with formula (3);

$$\therefore\ (D_t x)^2 = \frac{p^2}{q^2},\ \text{and}\ D_t p - D_t x \,.\, D_t q = 0.$$

There are two values of $D_t x$, we take the negative form as being the simpler:

$$\therefore\ D_t x = -\frac{p}{q},\ \text{and}\ D_t p + \frac{p}{q} \,.\, D_t q = 0;$$

$$\therefore\ D_t u = p + q D_t x = 0,\ \text{and}\ pq = F(\xi);$$

$$\therefore\ u = \xi;\ \therefore\ pq = F(u).$$

This is a first integral. It may be integrated as follows. Let θ be a new dependent variable, such that

$$p = \theta \sqrt{Fu},\ \text{and}\ \therefore\ q = \frac{1}{\theta} \,.\, \sqrt{Fu}.$$

Then, because

$$du = pdt + qdx,$$

$$\therefore\ \frac{du}{\sqrt{Fu}} = \theta dt + \frac{dx}{\theta};$$

$$\left.\begin{aligned} \therefore\ \phi(u) &= \theta t + \frac{x}{\theta} + f(\theta) \\ 0 &= t - \frac{x}{\theta^2} + f'(\theta) \end{aligned}\right\} \text{..................(1).}$$

The elimination of θ between these two equations will give the required integral.

This integral has been obtained from one of the values of $D_t x$; and this value has led us to the equation

$$F(u) = pq$$

as the corresponding first integral. It is easy to obtain the second first integral, if required, by employing the second value of $D_t x$. In this case we must introduce the symbol Δ_t, and the equations we have to deal with are

$$\Delta_t x = \frac{p}{q}, \text{ and } \Delta_t p - \Delta_t x \,.\, \Delta_t q = 0.$$

From these we can obtain two integrable equations. For eliminating $\Delta_t x$ we have

$$\Delta_t p - \frac{p}{q} \Delta_t q = 0;$$

$$\therefore\ \frac{p}{q} = \xi';$$

$$\therefore\ \Delta_t x = \xi';$$

$$\begin{aligned} \therefore\ x &= \xi' t + \phi(\xi') \\ &= \frac{p}{q} \,.\, t + \phi\left(\frac{p}{q}\right). \end{aligned}$$

This is the second first integral, and if we substitute θ^2 for its equal $\frac{p}{q}$, we shall see that this first integral constitutes the second of the two equations (1). The final integral derived from this equation will be found to be the same as that already found and expressed by the equations (1).

56. Hitherto we have been able to proceed to the final integrals of the Examples proposed without meeting with any impediment. But we shall not be able to do so in the remaining Examples of this Chapter. These will bring us face to face with difficulties which require careful consideration. But when they are resolved they seem to have constituted an essential part of the process of integration, because they have forced us out of the track of ordinary routine when ordinary routine was inadequate to the task of integration. They teach us that differential equations, which fall under this class, sometimes represent Geometrical properties or Physical phænomena which an ordinary integral (an integral which can be obtained in the ordinary way) cannot express. In fact the difficulties which force us out of the beaten track are necessary, because the integral which is to be found has to express results which cannot be expressed by any single integral. Two integrals are found to be necessary, each of which is independent of the other; and yet though capable of being solitary, they are to be also capable of being joined together by a single link. This language will seem mysterious at present, but it will be explained by the examples which follow.

An important question will also be suggested and in one instance answered,—must there necessarily be an integral of every partial differential equation that can be proposed? We may write down a partial differential equation at random, does it necessarily express possible geometrical or physical properties? and if it does not, ought we to expect to be able to find an integral of it? If the conditions and requirements of a problem be of such a nature that the variables cannot satisfy them without assuming the attributes of infinity, as in Example 10 of this Chapter, is it reasonable to expect that the differential equation which embodies them can admit of an integral?

There may be degrees of impossibility of this kind. As in ordinary Algebra there are formulæ which are true for one set of numbers (say, *odd* numbers) and untrue for any other set of numbers (say, *even* numbers); and other formulæ which are true for *even* numbers and untrue for *odd* numbers; so there may be two forms of integrals of the same equation, one of which may be true for one set of values of the variables, and the other for another set. How is it to be indicated in an *integral*, when there is no indication of it in the partial differential equation from which the integral was derived, that one of the forms of the integral is to be used in one set of circumstances, and in other circumstances the other?

The next equation we shall give for integration is derived from Ex. 1 by *arbitrarily* changing the algebraic sign of the middle term. The equation in Ex. 1 represents a family of Ruled Surfaces, what does the equation in Ex. 7 represent? Does it represent a possible property either in Geometry or Physics?

Ex. 7. To integrate

$$q^2d_t^2u + 2pqd_td_xu + p^2d_x^2u = 0.$$

Compare this, after dividing it by q^2, with formula (4);

$$\therefore\ D_tx = \frac{p}{q},\ \text{and}\ D_tp + D_tx\,.\,D_tq = 0\ \ldots\ldots\ldots\ldots(1).$$

These two combined give

$$qD_tp + pD_tq = 0;$$

$$\therefore pq = \xi.$$

No other integrable equation can be obtained from equations (1). Here then we meet with only partial success. What use can we make of this degree of success? It must be of some value. The following has suggested itself.

Let us now abandon the equations (1), and use our success as a hint to change the *dependent* variable from u to pq (or ξ). This change of the dependent variable being made, the proposed equation takes the form

$$d_t(pq) + \frac{p}{q} \cdot d_x(pq) = 0.$$

Now, on comparing this with the general formula,

$$D_t(pq) = d_t(pq) + D_tx \cdot d_x(pq)$$

we perceive that we must have

$$D_t(pq) = 0,$$

an equation which can be satisfied in two ways only, viz. by supposing

either $pq =$ constant; or $pq =$ quasi-constant.

The latter involves the unintegrable equation $D_tx = \frac{p}{q}$; but the former evades this equation. The only way of arriving at a final integral is therefore the supposition,

$$pq = C, \text{ a constant},$$

We consider the impossibility of forming *two* integrable equations out of equations (1) to be a necessary incident,—necessary to force us upon this alternative equation.

$$\begin{aligned} \therefore\ du &= pdt + qdx \\ &= pdt + \frac{C}{p}\,dx\,; \end{aligned}$$

$$\left.\begin{aligned} \therefore\ u &= pt + \frac{C}{p}\,x + F(p) \\ 0 &= t - \frac{C}{p^2}\,x + F'(p) \end{aligned}\right\}.$$

The elimination of p between these equations will give the only possible integral; which therefore contains *one arbitrary function, and one independent arbitrary constant.* According to the usual rule, there might be in a general integral *two* arbitrary functions, but in this Example the place of one of the arbitrary functions is supplied by an independent arbitrary constant.

Ex. 8. To integrate

$$d_t^2u + 2pd_td_xu + (p^2 - a^2)\,d_x^2u = 0.$$

Since $2p = (p+a) + (p-a)$, we may compare this equation with formula (5),

$$D_tp + MD_tq = d_t^2u + (D_tx + M)\,d_td_xu + MD_tx\,.\,d_x^2u\,;$$

$$\therefore \text{ either } D_tx = p + a, \text{ and } M = p - a\,;$$

$$\text{or } D_tx = p - a, \text{ and } M = p + a.$$

We begin with the former of these;

$$\therefore\ D_tx = p + a, \text{ and } D_tp + (p-a)\,D_tq = 0\,;$$

$$\therefore\ (p-a)\,\epsilon^q = \xi.$$

The equation $D_t x = p + a$ cannot be integrated; we therefore proceed as in the last Example. We take $(p-a)\epsilon^q$ for a new dependent variable, and this conducts us to the equation

$$(p-a)\epsilon^q = C, \text{ a constant.}$$

And this being integrated in the same manner as before, gives

$$\left.\begin{array}{l} u = (C\epsilon^{-q} + a)\,t + qx + F(q) \\ 0 = -C\epsilon^{-q}t + x + F'(q) \end{array}\right\} \ldots\ldots\ldots\ldots (1),$$

and if we eliminate q between these equations we shall have *a* general integral of the proposed equation.

We say *a* general because we have another set of equations to deal with, viz.

$$D_t x = p - a, \text{ and } M = p + a;$$

which are also included in the proposed equation. The resulting integral in this case is

$$\left.\begin{array}{l} u = (C'\epsilon^{-q} - a)\,t + qx + f(q) \\ 0 = -C'\epsilon^{-q}\,t + x + f'(q) \end{array}\right\} \ldots\ldots\ldots\ldots (2).$$

Thus we have in an independent manner obtained two integrals of equal authority. We have now to examine the significance of this circumstance, which it is obvious could not have occurred had an integral been obtainable in the ordinary manner. Let us examine the two integrals.

1. The two arbitrary constants C and C' are not necessarily equal. Nothing in the processes of integration necessitates their equality. So likewise the two functions F and f are not necessarily equal, being absolutely arbitrary.

2. Nevertheless, if we assume $t=0$, the two integrals become

$$\left.\begin{aligned} u &= qx + F(q) \\ 0 &= x + F'(q) \end{aligned}\right\} \text{.....................} (3),$$

and

$$\left.\begin{aligned} u &= qx + f(q) \\ 0 &= x + f'(q) \end{aligned}\right\} \text{.....................} (4),$$

which place before us the following alternative:

(α). Either F and f must be equal;

or (β). The two integrals cannot both be satisfied with the same values of x; that is, they cannot cover the same ground.

3. If F and f are equal, write in equations (2) F for f; and the same in equations (4).

Two circumstances will now be evident.

(γ). The equations (3) and (4) exactly agree; but equations (1) and (2) differ, and cannot be made to agree even by assuming $C' = C$.

(δ). The phænomena represented by the first and second integrals (1) and (2) exactly agree when $t=0$; but they disagree for all other values of t.

If for easiness of expression we call t time, and x place, we may express these results (γ), (δ), as follows:—

When F and f are equal, the two integrals (1), (2) express phænomena which had a common origin at the time $t=0$; but afterwards they become essentially distinct,—*solitary*, not existing in the same place. If t have the same value in both integrals, x cannot have the same value in both. The phænomena therefore which coexisted both in time and place at their origin afterwards become solitary,—each existing in its own place.

In their most general form the two integrals of equal authority which we have obtained represent phænomena which do not coexist in space, and are every way independent of each other. But in the restricted forms obtained by assuming F equal to f they represent phænomena which have no other connexion than this,—that they are due to a common originating cause.

The general inference therefore is,—that the proposed differential equation embodies properties or phænomena which may be either,—1st, absolutely independent, and capable of existing separately and alone; or, 2nd, dependent in their origin, but afterwards independent, and incapable of coexisting with the same values of co-ordinates t and x.

The Wave of Sound.

The two preceding Examples will in some measure have prepared us for entering on the integration of the dynamical equation for the transmission of the sound-wave through a thin straight horizontal tube of elastic medium.

Ex. 9. To integrate

$$d_t^2 u = \frac{a^2}{q^2} . d_x^2 u.$$

This is the dynamical equation for the motion of the sound-wave. On comparing it with formula (1), we find

$$(D_t x)^2 = \frac{a^2}{q^2}\,; \text{ and } D_t p - D_t x \,.\, D_t q = 0.$$

We have therefore two cases to consider arising from the two equations,

$$D_t x = \frac{a}{q}, \text{ and } \Delta_t x = -\frac{a}{q}.$$

We begin with the former of these. Both of them are unintegrable; but the former enables us to obtain the following integrable equation,

$$\therefore\ D_t p - \frac{a}{q} . D_t q = 0,$$

$$\therefore\ q\epsilon^{-\frac{p}{a}} = \xi.$$

On account of the impossibility of obtaining another integrable equation with the help of $D_t x = \frac{a}{q}$ on which the integral just obtained depends, we are obliged to take $q\epsilon^{-\frac{p}{a}}$ as the new dependent variable instead of u; and this leads to the equation,

$$q\epsilon^{-\frac{p}{a}} = C, \text{ a constant;}$$

whence we find, as in the preceding examples,

$$\left.\begin{aligned} u &= pt + C\epsilon^{\frac{p}{a}} x + F(p) \\ 0 &= t + \frac{C}{a}\epsilon^{\frac{p}{a}} x + F'(p) \end{aligned}\right\} \dots\dots\dots\dots\dots (1).$$

The elimination of p between these equations will give us one of the general integrals. The other will be found by writing $-a$ for a.

$$\left.\begin{aligned} \therefore\ u &= pt + C'\epsilon^{-\frac{p}{a}} x + f(p) \\ 0 &= t - \frac{C'}{a}\epsilon^{-\frac{p}{a}} x + f'(p) \end{aligned}\right\} \dots\dots\dots\dots\dots (2).$$

All the remarks that were made on the two general integrals of the last Example might be repeated here; but as in this example we know the meaning of the symbols t, x; and also know the origin of the proposed differential equation, we can use more definite language here than there.

The origin of the proposed differential equation is this:— Given a uniform elastic medium (as air) at rest. A sudden short disturbance occurs at a particular part of the medium. The details of the original disturbance are given. Find the state of the medium at any subsequent time. This is expressed in differential symbols in the proposed equation.

We know that a denotes the value of the fraction $\frac{\text{pressure}}{\text{density}}$; x the distance of a particle of air from a fixed section of the tube before it was disturbed; and u the corresponding distance of the same particle when in motion at the time t. Also $p = d_t u = v$ the velocity of the same particle at the same time, and

$$q = d_x u = \frac{\rho}{\rho_0},$$

where ρ_0 is the equilibrium density of every part of the medium, and ρ the density at the distance u and time t.

Hence, in reference to the first general integral (1), the equation,

$$q\epsilon^{-\frac{p}{a}} = C, \text{ becomes } \frac{\rho}{\rho_0}\epsilon^{-\frac{v}{a}} = C.$$

But in the forefront of a wave ρ becomes ρ_0 and $v = 0$, which values being written in the equation for ρ and v give

$$C = 1;$$

$$\therefore \text{ always } \rho = \rho_0 \epsilon^{\frac{v}{a}} \text{ ....(3).}$$

In a similar way we arrive in the case of the second integral at the equations

$$C' = 1, \text{ and } \rho = \rho_0 \epsilon^{-\frac{v}{a}} \dots\dots\dots\dots\dots\dots (4).$$

It is evident then that (3) and (4) cannot coexist in the same part of the tube, since (3) exists where ρ is greater, and (4) where it is less, than ρ_0.

Hence the two integrals (1) and (2) cannot both hold good at the same time in the same part of the tube.

Taken separately (supposing v positive) (1) represents a wave of condensation (a positive wave); and (2) a wave of rarefaction (a negative wave); and consequently such waves if they exist at the same time must be in different parts of the tube.

We shall now shew how two such waves may have a common origin, in which case they must both be embraced in a connected system of general integrals.

Let us suppose a rigid diaphragm in the middle of the tube, and let this diaphragm suddenly move forward through a short space. The result of this *one* movement will be the generation of *two* waves,—a positive wave on the front of the diaphragm, and a negative wave on the back of the diaphragm. In the strictest sense the two waves have a common origin, and they are alike included in the same differential equation of motion. It is the work of integration to bring to light the fact that one of them ever remains on one side of the diaphragm, and the other on the other side; and that the motion of one wave is in one direction, and that of the other in the opposite direction, while yet v has the same algebraic sign in both.

Now the motion of the diaphragm, or (which is the same thing) of the particles of air in contact with the diaphragm on both its sides, is the datum from which both waves are to be determined. If we take the original position of the diaphragm to be where $x=0$, we see that (since $C=C'=1$) for the particles in contact with the diaphragm, we have from equations (1), on the front side, where $x=0$,

$$\left.\begin{aligned} u &= pt+F(p) \\ 0 &= t+F'(p) \end{aligned}\right\},$$

and from equations (2), on the back side, where $x=0$,

$$\left.\begin{aligned} u &= pt+f(p) \\ 0 &= t+f'(p) \end{aligned}\right\}.$$

The comparison of these sets with one another shews us that $F=f$, since t and p (or v) are the same in both sets of equations.

Hence when a positive and a negative wave have a common genesis, the former will be represented by equations (1) and the latter by equations (2) when F is written for f. And when this is done, the two sets of equations (1) (2) taken together embrace the *whole* effect due to the motion of the diaphragm, and consequently for that case represent the complete integral of the proposed equation.

Here then we have a device by which Analytical Symbols meet all possible cases. We may have to consider the properties of a solitary wave, it may be a wave of condensation; or it may be a wave of rarefaction; we are furnished with appropriate integrals, which in themselves shew that they cannot be used together for the same part of the medium, corresponding to the physical fact that two such waves cannot exist together in the same place. Or we may have given, the fact that an agitation has been produced in one place in the cylinder, and we know that two waves moving in opposite directions will result from it. The originating cause is one, and the arbitrary function which has to represent that cause is one. The two integrals become connected by writing F for f; but still the integrals are *two*, and though connected in the way just mentioned cannot be combined together so as to be coexistent in the same part of the cylinder, all which corresponds accurately with physical fact.

Thus what at first seems to be a failure of integration is a necessary part of the process for obtaining the two integrals which may be taken separately; or when taken together constitute the complete integral of the proposed equation.

These ideas are the necessary results of the method of integration proposed in this Essay. They are put forward with diffidence for the consideration of mathematicians who have more leisure to devote to such difficulties than the author.

One thing is evident, that the existence of an accumulation of medium in one part of the cylinder necessitates the existence at the same time of an equal deficiency of medium in another part of the cylinder, since the motion of the diaphragm in the genesis of the two waves cannot alter the whole quantity of medium in the tube: and as these two effects of one cause are coexistent in time but not in place, two integrals seem to be inevitably necessary. If necessary, by what means is the process of integration to force them upon us? In the last two Examples we have seen how this may be done. No doubt Analytical Symbols may in other cases have other methods of accomplishing the same object.

Equal Radii of Curvature of Contrary Signs.

An example will now be given which will bring before us a difficulty of another kind. Mathematicians have investigated the partial differential equation which embodies the condition, that at each point of a surface the two principal radii of curvature are equal and of opposite signs. And this equation has been proposed for integration with the object of finding the general equation of the family of surfaces which all possess this property. This will be our next example.

Ex. 10. To integrate

$$(1+p^2)\, d_t^2u - 2pq d_t d_x u + (1+q^2)\, d_x^2 u = 0. \qquad q. \quad p$$

Divide by $1+p^2$ and compare the equation with formula (5).

$$\therefore\ D_t x + M = -\frac{2pq}{1+p^2}, \text{ and } M.D_t x = \frac{1+q^2}{1+p^2};$$

$$\text{and } D_t p + M D_t q = 0.$$

The first two of these equations give us two values for $D_t x$, and two corresponding values for M. But neither of the values of $D_t x$ will enable us to obtain an integrable equation. It is not, as in the preceding Examples, that we are furnished with one integrable and one unintegrable equation. In this example nothing integrable can be found. ?

Let us endeavour to find out the reason of this difficulty.

$q - cp \pm \sqrt{1+c^2}\,\sqrt{}1$

If at any point of a surface C_1, C_2 represent the principal curvatures, and C any other curvature at the same point, we have the following equation, connecting these quantities together, in Treatises on the Geometry of Surfaces,

$$C = C_1 \cos^2\theta + C_2 \sin^2\theta.$$

The meaning of θ is well known. Now if $C_1 = -C_2$, which is the case in this example, we have

$$C = C_1 \cos 2\theta.$$

And this equation shews that

as θ passes from -45° to $+45^\circ$, C is $+$;
.... $+45^\circ$ to 135°, C is $-$;
.... 135° to 225°, C is $+$;
.... 225° to 315°, C is $-$.

Thus at a point of a surface where the principal radii of curvature are equal and of contrary signs, the surface will be corrugated, the corrugations tending *from* the point as a centre, and being alternately above and below the tangent plane at the point. Briefly we may say, such a point is *a centre of corrugation*. But from the very nature of a continuous surface *every* point of it cannot be a centre of corrugation; and consequently there is no surface that possesses the property embodied in the proposed differential equation. Are we not correct in expecting that such an equation cannot have an integral?

Yet there is one way of finding an integral,—to assume that simultaneously we must have

$$d_t^2 u = 0,\quad d_t d_s u = 0,\quad d_s^2 u = 0.$$

These satisfy the proposed differential equation and give us the following integral,

$$u = At + Bx + C,$$

which is the equation of a plane surface, and is in fact the only answer the proposed problem admits of.

Strictly speaking, however, this integral is neither an answer to the proposed problem, nor an integral of the proposed equation. For a plane surface has no curvature, and consequently also no radii of curvature: and the three equations $d_t^2u = 0$, $d_td_xu = 0$, $d_x^2u = 0$ are an evasion of the proposed equation, not an equivalent for it.

CHAPTER XI.

SUPPLEMENTAL ARTICLES.

In the preceding Chapters the forms of quasi-constants have been determined by comparison of a proposed equation with general formulæ; and this has been found sufficient for the majority of equations generally proposed for integration in the ordinary Works upon this subject. But the principle on which the *Method* of this Essay is founded is embodied in the right use of the equation

$$D_t u = d_t u + D_t x \,.\, d_x u + D_t y \,.\, d_y u + \dots$$

or, more generally,

$$D_\tau u = D_\tau t \,.\, D_t u + D_\tau x \,.\, d_x u + D_\tau y \,.\, d_y u + \dots$$

The use of other formulæ is a convenience in most cases, but not a necessity: and there are equations and classes of equations, the successful integration of which will depend solely on a skilful use of the above fundamental form whereby the change of the system of independent variables is effected.

To the skilful use of this form we must look for an extension of success into fresh fields of difficulties. An example or two will illustrate what is here meant.

I. Ex. 1. To integrate

$$x = \frac{d_t u}{d_x u}\,t + \phi\left(\frac{d_t u}{d_x u}\right).$$

None of our formulæ are of this form, we therefore arbitrarily assume that the new system of variables is such as to allow us to write

$$D_t x = -\frac{d_t u}{d_x u}.$$

The reason of the minus sign being assumed in this equation will be seen in the next step,—it gives us an integrable equation.

$$\therefore\ d_t u + D_t x \,.\, d_x u = 0 = D_t u;$$

$$\therefore\ u = \xi.$$

For brevity write v for $\frac{d_t u}{d_x u}$ in the proposed equation,

$$\therefore\ x = vt + \phi(v) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Differentiate this equation with D_t, remembering that $D_t x = -v$;

$$\therefore\ 2v + tD_t v + \phi'(v) \,.\, D_t v = 0.$$

Divide by $\sqrt{v}$ and integrate with S_t;

$$\therefore\ 2t\sqrt{v} + s_v\left(\frac{\phi' v}{\sqrt{v}}\right) = F(\xi) \ldots\ldots\ldots\ldots (2).$$

Substituting u for ξ and eliminating v between equations (1) and (2), we have the integral required.

The above is applicable to two independent variables only, but the following example is so stated as to be applicable to any number of independent variables, D_tu being equal to $d_tu + D_tx \,.\, d_xu + D_ty \,.\, d_yu + \dots$; and the quantity represented by Q is under no restriction but this, that D_tQ is to be either a constant or a quasi-constant.

Ex. 2. To integrate

$$Q + uD_tu = \phi(D_tu).$$

For brevity write v for D_tu,

$$\therefore\; Q + uv = \phi(v) \dots\dots\dots\dots (1).$$

Divide this by v and differentiate with D_t, remembering that $D_tu = v$;

$$\therefore\; \frac{D_tQ + v^2}{v} - \frac{Q}{v^2} \,.\, D_tv = D_t\left(\frac{\phi v}{v}\right).$$

Divide by $\sqrt{D_tQ + v^2}$ and integrate with S_t;

$$\therefore\; \left(\frac{D_tQ}{v^2} + 1\right)^{\frac{1}{2}} \,.\, \frac{Q}{D_tQ} + S_t\,(D_tQ + v^2)^{-\frac{1}{2}} D_t\left(\frac{\phi v}{v}\right) = F(\xi, \eta, \zeta, \dots);$$

$$\text{or, } \left(\frac{D_tQ}{v^2} + 1\right)^{\frac{1}{2}} \,.\, \frac{Q}{D_tQ} + S_v\,(D_tQ + v^2)^{-\frac{1}{2}}\, d_v\left(\frac{\phi v}{v}\right) = F(\xi, \eta, \zeta, \dots).$$

We must substitute for $\xi, \eta, \zeta \dots$ their values and then eliminate v between this and equation (1); and the result will be the required integral.

The following are particular examples of this class.

(3) To integrate

$$ht + kx + u\,(d_tu + ad_xu) = \phi\,(d_tu + ad_xu).$$

(4) To integrate

$$ht + kx + u\left(d_t u + \frac{x}{t} d_x u\right) = \phi\left(d_t u + \frac{x}{t} d_x u\right).$$

(5) To integrate

$$ht + kx + ly + u\left(d_t u + \frac{x}{t} d_x u + a d_y u\right) = \phi\left(d_t u + \frac{x}{t} d_x u + a d_y u\right).$$

II. The chief object in Chap. VIII. was to shew how to deprive a particular class of linear equations of coefficients which were not constant, in order to reduce them to linear equations with constant coefficients. These being always integrable, we were enabled to ascend from their integrals to those of the proposed equations. But in Chap. VII. we were often conducted for the value of u to a result of one of the forms following:

$$(\phi d_t)^n . u = 0, \quad \text{or} \quad (\phi d_t)^n t (\phi d_t)^{1-n} u = 0.$$

It is desirable to exhibit the direct dependence of u on v; v being an auxiliary quantity defined by the equation

$$(\phi d_t) v = 0 \text{..........................} (1).$$

By actual integration we have

$$s_t^n 0 = A_1 + A_2 t + A_3 t^2 + \ldots\ldots + A_n t^{n-1};$$

that is, $s_t^n 0$ will represent a series in integer powers of t from 0 to $n-1$, the coefficients of which are all *arbitrary*. We shall therefore take $s_t^n 0$ to represent any series of the above form when all its coefficients are arbitrary. It is to be distinctly understood that the series represented by $s_t^n 0$ is *definite* in form and *indefinite* in the coefficients of its terms. This will give rise to a peculiarity in its use,—it will enable us with proper explanations to use the sign of *equality* in some cases where the strict logic of the steps would require the use of a sign of *equivalence* only.

Since s_t0 is constant, by multiplying equation (1) by it we have

$$(\phi d_t) \,.\, vs_t0 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, since $s_t^2 0$ contains t to the first power only, if we operate with (ϕd_t) on $vs_t^2 0$, by formula (1) Art. 39 we have

$$(\phi d_t) \,.\, vs_t^2 0 = s_t^2 0 \,.\, (\phi d_t) v + s_t 0 \,.\, (\phi' d_t) v$$

$$= (\phi' d_t) \,.\, vs_t 0, \quad \text{by (1);}$$

$$\therefore\ (\phi d_t)^2 \,.\, vs_t^2 0 = (\phi' d_t)\,(\phi d_t) \,.\, vs_t 0$$

$$= 0, \quad \text{by (2).}$$

We see then that the formula

$$(\phi d_t)^n \,.\, vs_t^n 0 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

holds good for $n = 1$, and $n = 2$. We see also that it holds good in these cases because v satisfies equation (1); and that in consequence of this we may write for v any quantity whatever (or the sum of any such quantities) which satisfies equation (1). The form (3) will therefore hold good for $n = 1, 2$, if we write for v any quantity such as $(\phi' d_t) v$, or $(\phi'' d_t) v$, ... because each of these when written for v in equation (1) will satisfy it as well as v.

We will now suppose that equation (3) is true for all integer values of n from 1 to $(n-1)$; and we shall shew that in that case it is true for the next value of n.

By the formula of Art. 36 we have

$$(\phi d_t) . vs_t^n 0 = s_t^n 0 . (\phi d_t) v + s_t^{n-1} 0 . (\phi' d_t) v + \frac{1}{1.2} s_t^{n-2} 0 . (\phi'' d_t) v + \ldots$$

$$= s_t^{n-1}0 \,.\, (\phi' d_t) v + \frac{1}{1\,.\,2} s_t^{n-2}0 \,.\, (\phi'' d_t)\, v + \dots\dots$$

But each term of the right hand will vanish, by hypothesis, if it be operated on with $(\phi d_t)^{n-1}$;

$$\therefore\ (\phi d_t)^n \,.\, v s_t^n 0 = 0 \ \dots\dots\dots\dots\dots \ (3).$$

This equation is therefore universally true for positive integer values of n, both for v and for any quantity instead of v which will satisfy equation (1).

Now if equation (1) be a linear equation of the m^{th} order, we know that the complete value of v will consist of the sum of m independent partial values, each of which will satisfy equation (1); and consequently the left-hand member of equation (3) will be made up of the sum of m parts, each of which independently satisfies it.

Now a glance at the process by which equation (3) has been proved will convince us that it is not necessary that the arbitrary constants in $s_t^n 0$ should have the same values in each of the independent parts of v. We therefore interpret the symbol $v s_t^n 0$ as meaning that *each part of* v is to be multiplied by $s_t^n 0$; and that $s_t^n 0$ is to be considered in each of the parts to be independently arbitrary.

From this explanation it will be seen that if $(\phi d_t) \,.\, v = 0$ be of the m^{th} order, v involves m independent partial integrals; and, as $s_t^n 0$ involves n arbitrary constants which are different for each part of v, the expression $v s_t^n 0$ will involve mn arbitrary constants.

If, therefore, we have such an equation as the following,

$$(\phi d_t)^n \,.\, u = 0 \ \dots\dots\dots\dots\dots\dots\dots \ (4),$$

by comparing it with equation (3) we have the following as its integral,

$$u = vs_t^n 0 \dots\dots\dots\dots\dots\dots\dots\dots (5),$$

and this is the *complete* integral of equation (4), because it contains mn arbitrary constants, and equation (4) is of the degree mn.

We are now able also to bring u in the equation,

$$(\phi d_t)^n t (\phi d_t)^{1-n} u = 0 \dots\dots\dots\dots\dots\dots (6),$$

into connexion with v; for on comparing this equation with equation (4), we have here $t(\phi d_t)^{1-n}u$ for u there; and consequently by equation (5) we have

$$t(\phi d_t)^{1-n} u = vs_t^n 0 \dots\dots\dots\dots\dots\dots (7).$$

Now in dividing this equation by t we must remember that $s_t^n 0$ is a series in integral powers of t from t^0 to t^{n-1}; and therefore, if we divide it by t, the quotient will consist of two parts, one of which contains powers of t from t^0 to t^{n-2}, and which may therefore be represented by $s_t^{n-1}0$; and the other will be represented by $t^{-1}s_t 0$;

$$\therefore \ (\phi d_t)^{1-n} u = vs_t^{n-1} 0 + \frac{v}{t} s_t 0.$$

Operate on both sides of this equation with $(\phi d_t)^{n-1}$, {see equation (3)},

$$\therefore \ u = (\phi d_t)^{n-1} . \frac{v}{t} s_t 0 \dots\dots\dots\dots\dots\dots (8).$$

This is the integral of equation (6).

REMARK. $s_t 0$ will (as before explained) be different for every part of v; hence the value of u just found will contain m arbitrary constants if $(\phi d_t) v$ be of the m^{th} order.

It may also be remarked, that we are at liberty to write D_t for d_t; in which case

$$\phi(D_t)\,v = 0$$

will denote a linear partial differential equation of the m^{th} order, with constant or quasi-constant coefficients, and of any number of variables. Also, arbitrary constants will then be replaced in v by arbitrary functions of the quasi-constants.

III. To integrate $td_t^2u + \left(a + \frac{1}{2}\right) d_tu = \frac{1}{4} b^2u.$

Let $t = \tau^2$, and change the independent variable from t to τ;

$$\therefore\ \tau(d_\tau^2 - b^2)\,u + a\,.\,2d_\tau u = 0.$$

Operate on this with $(d_\tau^2 - b^2)^{-a}$, (see Ex. 9, Chap. VII.);

$$\therefore\ \tau(d_\tau^2 - b^2)^{1-a}u = (d_\tau^2 - b^2)^{-a}0$$
$$= (d_\tau - b)^{-a}0 + (d_\tau + b)^{-a}0 \ldots\ldots \text{(Art. 16)}.$$

As the two parts of u are independent, we may take them separately and add the results together for the complete value of u;

$$\therefore\ \tau(d_\tau - b)^{1-a}(d_\tau + b)^{1-a}u = (d_\tau - b)^{-a}0\,;$$
$$\therefore\ \tau\,.\,\epsilon^{b\tau}d_\tau^{1-a}\epsilon^{-b\tau}(d_\tau + b)^{1-a}u = \epsilon^{b\tau}d_\tau^{-a}\epsilon^{-b\tau}0,$$
$$= \epsilon^{b\tau}s_\tau^{a}0.$$

If now we divide by τ, and consider that s_τ^a0 contains powers of τ from τ^0 to τ^{a-1}, and that all the coefficients are arbitrary, we shall see that although

$$\tau^{-1}s_\tau^{a}0 \text{ is not } = s_\tau^{a-1}0 + \tau^{-1}s_\tau 0,$$

the two members of this inequality are *equivalent* for the purpose before us.

$$\therefore\ \epsilon^{b\tau} d_\tau{}^{1-a} \epsilon^{-b\tau} (d_\tau + b)^{1-a} u = \epsilon^{b\tau} s_\tau{}^{a-1} 0 + \epsilon^{b\tau} \tau^{-1} s_\tau 0,$$

$$\therefore\ (d_\tau + b)^{1-a} u = \epsilon^{b\tau} d_\tau{}^{a-1} \tau^{-1} s_\tau 0.$$

The right-hand member, though not equal to $\epsilon^{b\tau} \tau^{-a} s_\tau 0$, is equivalent to it, because $s_\tau 0$ is arbitrary. By actual differentiation, and preserving the arbitrary character of $s_\tau 0$, we have

$$\epsilon^{-b\tau} d_\tau{}^{1-a} \epsilon^{b\tau} u = \epsilon^{b\tau} \tau^{-a} s_\tau 0\,;$$

$$\therefore\ u = A \epsilon^{-b\tau} d_\tau{}^{a-1} (\epsilon^{2b\tau} \tau^{-a}),$$

A being an arbitrary constant.

The other part of u will be obtained from this by writing $-b$ for b;

$$\therefore\ u = A \epsilon^{-b\tau} d_\tau{}^{a-1} (\epsilon^{2b\tau} \tau^{-a}) + B \epsilon^{b\tau} d_\tau{}^{a-1} (\epsilon^{-2b\tau} \tau^{-a}).$$

After the differentiations have been performed we are to write $\sqrt{t}$ for τ.

IV. To integrate $t d_t^2 u + \left(\frac{1}{2} - a\right) d_t u = \frac{1}{4} b^2 u$.

Change the variable as in the preceding Article,

$$\therefore\ \tau\,(d_\tau^2 - b^2)\, u = 2a d_\tau u \ldots\ldots\ldots\ldots\ldots(1).$$

Operate on this with $(d_\tau^2 - b^2)^a$,

$$\therefore\ \tau (d_\tau^2 - b^2)^{a+1} u = 0\,;$$

$$\begin{aligned} \therefore\ u &= (d_\tau - b)^{-a-1} 0 + (d_\tau + b)^{-a-1} 0 \\ &= \epsilon^{b\tau} s_\tau{}^{a+1} \epsilon^{-b\tau} 0 + \epsilon^{-b\tau} s_\tau{}^{a+1} \epsilon^{b\tau} 0 \\ &= \epsilon^{b\tau} s_\tau{}^{a+1} 0 + \epsilon^{-b\tau} s_\tau{}^{a+1} 0. \end{aligned}$$

The two parts of u are independent. Each part contains $a + 1$ arbitrary constants; and, as there ought only to be one in each part, the supernumeraries are to be determined by substitution in equation (1). Taking one of the two parts of

u and writing $\epsilon^{-b\tau}s_\tau^{a+1}0$ for u in equation (1), and assuming

$$s_\tau^{a+1}0 = B_0 + B_1\tau + B_2\tau^2 + \ldots + B_a\tau^a,$$

we find the following to be the law of the coefficients,

$$B_{k+1} = \frac{2b}{k+1} \cdot \frac{a-k}{2a-k} \cdot B_k.$$

If in the other term of u we assume

$$s_\tau^{a+1}0 = A_0 + A_1\tau + A_2\tau^2 + \ldots + A_a\tau^a,$$

we have merely to write $-b$ for b in the above law;

$$\therefore\ A_{k+1} = -\frac{2b}{k+1} \cdot \frac{a-k}{2a-k} \cdot A_k;$$

$$\therefore\ u = \epsilon^{b\tau}(A_0 + A_1\tau + A_2\tau^2 + \ldots + A_a\tau^a)$$
$$+ \epsilon^{-b\tau}(B_0 + B_1\tau + B_2\tau^2 + \ldots + B_a\tau^a);$$

and thus the integral required is completely determined. It only remains to write in this result $\sqrt{t}$ for τ.

V. In the preceding Chapters we have spoken of the equation $d_t d_x u + cu = 0$ as in general unintegrable in finite terms; and in Art. 27 we have shewn that its integral, if it admit of one in finite terms, cannot contain arbitrary functions. We shall proceed to shew how a complete *first* integral of it can always be obtained; and also that with a certain restriction the *final* integral can be obtained also in finite terms.

For, since

$$d_t d_x u + cu = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

multiply by t and differentiate with d_t, having first commutated d_t, d_x;

$$\therefore\ d_t d_x \,.\, t d_t u + c t d_t u + cu = 0,$$

$$\text{or}\ \ (d_t d_x + c) \,.\, t d_t u + cu = 0.$$

Similarly we obtain

$$(d_t d_x + c) . x d_x u + cu = 0;$$

$$\therefore\ (d_t d_x + c)(t d_t u - x d_x u) = 0,$$

which being compared with equation (1) gives, by integrating with $(d_t d_x + c)^{-1}$,

$$t d_t u - x d_x u = ku \ldots\ldots\ldots\ldots (2).$$

This integral has been obtained in a general manner, and is therefore entitled to be taken as a first integral of the proposed equation. It contains no arbitrary function but only an arbitrary constant.

(1) Let us now eliminate d_x between (1) and (2);

$$\therefore\ t d_t^2 u + (1 - k) d_t u + cxu = 0 \ldots\ldots\ldots\ldots (3).$$

Here we see the proposed equation, which is of two independent variables, reduced to an equivalent equation of one independent variable, for x is here a quasi-constant. Unfortunately this equation is of a form that does not admit of general integration for all possible values of k, k being arbitrary. If, however, we refer to Arts. III. and IV. we shall find it there proved that it does admit of integration in finite terms when k differs from $\frac{1}{2}$ by a positive or negative integer; and that the simplest or root-form of the integral is that which corresponds to $k = \frac{1}{2}$.

Now, by Art. III., if U be the integral for this value of k,

$$U = A\epsilon^{-b\tau} s_\tau \epsilon^{2b\tau} + B\epsilon^{b\tau} s_\tau \epsilon^{-2b\tau};$$

and this is equivalent to

$$U\sqrt{x} = A\epsilon^{2\sqrt{-cxt}} + B\epsilon^{-2\sqrt{-cxt}} \ldots\ldots\ldots\ldots (4),$$

since $b^2 = -4cx$, and $\tau = \sqrt{t}$. This is only one part of the integral, for in obtaining this equation we treated t as the leading independent variable. Had we taken x instead of t we should have found,

$$U\sqrt{t} = A'\epsilon^{2\sqrt{-cxt}} + B'\epsilon^{-2\sqrt{-cxt}}.$$

Consequently the complete integral, found by taking the sum of the partial values of u, will be

$$U = \left(\frac{A}{\sqrt{x}} + \frac{A'}{\sqrt{t}}\right)\epsilon^{2\sqrt{-cxt}} + \left(\frac{B}{\sqrt{x}} + \frac{B'}{\sqrt{t}}\right)\epsilon^{-2\sqrt{-cxt}} \text{.......}(5).$$

This is the complete finite root-integral of the proposed equation.

We call it the root-integral because the integrals for all other values of k which differ by integers from $\frac{1}{2}$ can be obtained from this by mere integration or differentiation with s_t or d_t.

For U in (4) satisfies the equation (found from (3) by writing $\frac{1}{2}$ for k),

$$td_t^2 U + \frac{1}{2} d_t U + cxU = 0;$$

operate on (3) with $d_t^{k-\frac{1}{2}}$;

$$\therefore\ td_t^2 \,.\, d_t^{k-\frac{1}{2}} u + \frac{1}{2} d_t \,.\, d_t^{k-\frac{1}{2}} u + cx \,.\, d_t^{k-\frac{1}{2}} u = 0.$$

Comparing these two equations, we see that

$$d_t^{k-\frac{1}{2}} u = U;$$

$$\therefore\ u = s_t^{k-\frac{1}{2}} U.$$

Thus u can be derived from U exactly in the same way as in all linear equations with constant coefficients. The arbitrary constant k in this case points only to the fact that if U be an integral, then $d_t^n U$ or $s_t^n U$ will be integrals also.

We may therefore take as the root-integral that which is the simplest in form, which is that given in (5) and corresponds to $k=\frac{1}{2}$. From that we may at pleasure derive other forms corresponding to different values of k by integration or differentiation.

(2) We have called equation (2) a *first* integral of the proposed equation, but it is so in a peculiar sense only. It was obtained by first raising the proposed equation an order higher, and then lowering it two orders by integration with the compound symbol $(d_t d_x + c)^{-1}$. This method of obtaining an integral always furnishes a result which requires to be checked and limited by substitution in the proposed or some equivalent equation, since on account of having been obtained from an equation of too high an order it may produce a result which is too general.

(3) Though we cannot for all possible values of the arbitrary constant k obtain an integral of equation (3), the equation (2) admits of general integration, giving

$$ux^k = F(tx) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

If therefore (2), the equation from which this is derived, be general, this is the form of the integral for all values of k. Nothing remains but to find the form of F for any proposed value of k. If u taken from this be substituted in (1), it produces an equation of the second order of the same form as (3); so that the root-form of F can only be found when $k=\frac{1}{2}$. We learn however from (6), that the subject of F is the same for all values of k.

(4) We must now draw attention to the circumstance that the proposed equation is known to be satisfied by the following equation,

$$u = A\epsilon^{ht-kx} + B\epsilon^{kx-ht} \text{ ....} (7),$$

h, k being arbitrary constants subject to the condition

$$hk = c.$$

Now this value of u does not satisfy (2); and consequently (2), which in one direction embraces too much, in another direction does not embrace enough. By raising the proposed equation to the third order we expected to embrace within our results more than belonged to us, and we now find that we have lost some results which ought to be included in a general integral. We shall recur to this subject in Art. XII.

VI. The equation discussed in the preceding Article is a particular case of the more general equation

$$\phi(d_t d_x)\, u = 0 \text{ ....} (1).$$

We shall shew that a first integral of this equation can be obtained.

By formula (2) of Art. 39 we have

$$\phi(d_t d_x) \,.\, t d_t u = t\phi(d_t d_x) \,.\, d_t u + \phi'(d_t d_x) \,.\, d_t d_x u.$$

Now $d_t u$ if written for u in equation (1) will satisfy it,

$$\therefore\ \phi(d_t d_x) \,.\, d_t u = 0;$$

$$\therefore\ \phi(d_t d_x) \,.\, t d_t u = \phi' . (d_t d_x) \,.\, d_t d_x u.$$

Similarly,

$$\phi (d_t d_x) \, . \, x d_x u = \phi' (d_t d_x) \, . \, d_t d_x u \, ;$$

$$\therefore \; \phi (d_t d_x) \, (t d_t u - x d_x u) = 0 :$$

which being compared with equation (1) gives, integrating this with $(\phi \, . \, d_t d_x)^{-1}$,

$$t d_t u - x d_x u = ku \text{ .... (2).}$$

This is therefore a first integral of the whole class of equations comprehended in equation (1).

The integral of the equation (2) is

$$u = t^k F (tx) \text{ .... (3).}$$

And this must be substituted in equation (1); and the resulting equation, if it can be integrated, will give the form of F and any restrictions on the values of k which may be necessary. It teaches us that the *subject* of F must be tx.

VII. There is another interesting equation, viz.

$$d_t d_x u + d_t d_y u + d_x d_y u = 0 \text{ .... (1),}$$

of which a first integral may be obtained in a similar way.

By commutation of the symbols of operation it will take the form

$$(d_x + d_y) \, . \, d_t u + d_x d_y u = 0.$$

Multiply by t and differentiate with d_t,

$$\therefore \; d_t (d_x + d_y) \, . \, t d_t u + d_x d_y \, . \, t d_t u + d_x d_y u = 0 \, ;$$

$$\therefore \; (d_t d_x + d_t d_y + d_x d_y) \, . \, t d_t u + d_x d_y u = 0.$$

Similarly,

$$(d_t d_x + d_t d_y + d_x d_y) \,.\, x d_x u + d_t d_y u = 0,$$

and

$$(d_t d_x + d_t d_y + d_x d_y) \,.\, y d_y u + d_t d_x u = 0.$$

Adding the last three equations together and taking account of equation (1), we have

$$(d_t d_x + d_t d_y + d_x d_y)\,(t d_t u + x d_x u + y d_y u) = 0,$$

and integrating this with $(d_t d_x + d_t d_y + d_x d_y)^{-1}$, we have

$$t d_t u + x d_x u + y d_y u = ku \text{ .... } (2).$$

This is a first integral of equation (1); and being integrated it gives

$$u = t^k f\left(\frac{x}{t}, \frac{y}{t}\right) \text{........................} (3)$$

as the integral of equation (1). But the form of f will have to be determined by the substitution of this value of u in equation (1). It will depend on the integration of an equation of *two* independent variables.

VIII. As a companion Article to VI. we may shew how to obtain a first integral of another important equation,

$$\phi\,(D_t \Delta_t)\, u = 0 \text{ ....} (1),$$

in which we suppose $D_t x = a$, and $\Delta_t x = b$; and $\therefore\ \xi = x - at$ and $\xi' = x - bt$.

By formula (3) of Art. 39 we have (since $\Delta_t \xi = b - a$, and $D_t \xi' = a - b$)

$$\begin{aligned} \phi\,(D_t \Delta_t) \,.\, \xi \Delta_t u &= \xi \,.\, \phi\,(D_t \Delta_t)\, \Delta_t u + (b - a)\, D_t \Delta_t u \\ &= (b - a)\, D_t \Delta_t u. \end{aligned}$$

Similarly,

$$\phi(D_t\Delta_t) \,.\, \xi' D_t u = (a-b)\, D_t \Delta_t u\,;$$

$$\therefore\ \phi(D_t\Delta_t) \,.\, (\xi\Delta_t u + \xi' D_t u) = 0\,;$$

$$\therefore\ \xi\Delta_t u + \xi' D_t u = ku \text{ ....(2).}$$

This is the first integral of equation (1). Its integral is

$$u = \xi^{\frac{k}{b-a}} F(\xi\xi')\,; \text{ or } \xi'^{\frac{k}{a-b}} f(\xi\xi') \text{............(3).}$$

These two forms of u are mutually equivalent. The form of the arbitrary function F or f, and any necessary restrictions on the values of k, will have to be ascertained by substitution in the proposed equation (1).

IX. If in the equation

$$d_t^2 u = d_x^2 u + cu \text{ ....(1),}$$

we change the independent variables by writing $t+x$ for t, and $t-x$ for x, it becomes

$$d_t d_x u = cu \text{ ....(2),}$$

the root-integral of which we know. We can therefore pass at once from the root-integral of this equation to the corresponding integral of equation (1) by writing $\frac{1}{2}(t+x)$ for t, and $\frac{1}{2}(t-x)$ for x. Consequently the root-integral of (1) is

$$Vu = A\epsilon^{\sqrt{(t^2-x^2)\,c}} + B\epsilon^{-\sqrt{(t^2-x^2)\,c}} \text{ ....(3),}$$

V being equal to either of the quantities

$$\sqrt{t+x}, \text{ and } \sqrt{t-x}.$$

Again: from this we may obtain the root-integral of the equation

$$d_t^2 u + d_x^2 u = cu \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

by writing $x\sqrt{-1}$ for x. Consequently the root-integral of (4) is

$$Vu = A\epsilon^{\sqrt{(t^2+x^2)\,c}} + B\epsilon^{-\sqrt{(t^2+x^2)\,c}} \;\ldots\ldots\ldots\ldots (5),$$

V being equal to either of the quantities,

$$\sqrt{(t + x\sqrt{-1})}, \text{ and } \sqrt{(t - x\sqrt{-1})}.$$

Remark. Both in (3) and (5) there are two values of u arising from the double values of V, and we may in each case take the sum of the two values of u for the complete integral.

X. To integrate the equation

$$d_t^2 u = d_x^2 u + d_y^2 u.$$

This may be written in the two forms

$$d_t^2 u = d_x^2 u + cu \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where c stands for d_y^2;

$$d_t^2 u = d_y^2 u + cu \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where c stands for d_x^2.

The latter may be deduced from the former by interchanging x and y.

From Art. IX., writing d_y for $\sqrt{c}$ in (3) we obtain

$$V_1 u = A\epsilon^{\sqrt{t^2-x^2}\,d_y} + B\epsilon^{-\sqrt{t^2-x^2}\,d_y}.$$

But in the case before us A and B are functions of y, being constants only in reference to d_t and d_x.

Now $$\epsilon^{\sqrt{t^2-x^2}\,d_y} F(y) = F(y + \sqrt{t^2 - x^2}),$$

and $$\epsilon^{-\sqrt{t^2-x^2}\,d_y} f(y) = f(y - \sqrt{t^2 - x^2});$$

$$\therefore\ V_1 u = F(y + \sqrt{t^2 - x^2}) + f(y - \sqrt{t^2 - x^2}) \ldots\ldots (3);$$

V_1 being equal to either of the two quantities,

$$\sqrt{t+x}, \text{ and } \sqrt{t-x}.$$

To obtain the value of u corresponding to the form (2) of the proposed equation, interchange x and y,

$$\therefore\ V_2 u = F_1(x + \sqrt{t^2 - y^2}) + f_1(x - \sqrt{t^2 - y^2}) \ \ldots (4),$$

V_2 being equal to either of the two quantities,

$$\sqrt{t+y}, \text{ and } \sqrt{t-y}.$$

The complete integral will be found by adding together the partial values of u.

Again; from the integral of

$$d_t^2 u = d_x^2 u + d_y^2 u,$$

we may deduce that of Laplace's equation,

$$d_t^2 u + d_x^2 u + d_y^2 u = 0,$$

by writing $x\sqrt{-1}$ for x, and $y\sqrt{-1}$ for y. The constituent parts of its integral may be written in the following forms;

$$V_1 u = F(\sqrt{t^2 + x^2} + y\sqrt{-1}) + f(\sqrt{t^2 + x^2} - y\sqrt{-1});$$

$$V_2 u = F_1(\sqrt{t^2 + y^2} + x\sqrt{-1}) + f_1(\sqrt{t^2 + y^2} - x\sqrt{-1}).$$

$$V_1 = (t + x\sqrt{-1})^{\frac{1}{2}}, \text{ or } (t - x\sqrt{-1})^{\frac{1}{2}};$$

$$V_2 = (t + y\sqrt{-1})^{\frac{1}{2}}, \text{ or } (t - y\sqrt{-1})^{\frac{1}{2}}.$$

The same remark respecting the complete value of u may be made in this Article as in the preceding one.

XI. If we have an equation of the form

$$d_t^2u = d_x^2u + d_y^2u + d_z^2u + \ldots + cu \ldots\ldots\ldots\ldots (1),$$

the number of independent variables being n, we can reduce it to another of $n-1$ independent variables. And this we can do with $n-1$ independent results.

For brevity use C to represent $d_y^2 + d_z^2 + \ldots + c$, and write the equation in the form

$$d_t^2u - d_x^2u = Cu \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If (as in Art. IX.) we write in this equation $t+x$ for t, and $t-x$ for x, it becomes

$$d_td_xu = Cu \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

In the manner of Art. V. we find from this equation

$$td_tu - xd_xu = ku \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

By eliminating d_x between this and (3), we obtain

$$td_t^2u + (1-k)\,d_tu = Cxu \ldots\ldots\ldots\ldots\ldots (5),$$

in which x is a quasi-constant. Consequently this equation contains only $n-1$ independent variables; and it is equivalent to the proposed equation.

Thus it appears that by the above process we can render any one of the independent variables $x, y, z, \ldots$ in equation (1) a quasi-constant; and consequently there are $n-1$ ways of reducing their number from n to $n-1$; and there are also $n-1$ independent equations of the form of equation (4).

Again; if we confine ourselves to the obtaining of the root-integrals of equation (1) we are to assume $k=\frac{1}{2}$ in equation (5); and if we then write τ^2 for $4tx$ this equation will take the form

$$d_\tau^2 u = Cu,$$

or

$$d_\tau^2 u = d_y^2 u + d_z^2 u + \ldots + cu;$$

which form is the same as that of the proposed equation deprived of one of its terms.

Thus by reversing the process we can pass from the form

$$d_t^2 u = cu, \text{ to } d_t^2 u = d_x^2 u + cu;$$

and from the last to

$$d_t^2 u = d_x^2 u + d_y^2 u + cu;$$

and so on to a similar equation of any number of independent variables.

XII. The following equation is interesting as throwing some light on the nature of the peculiar first integral obtained in some of the preceding Articles, and on which indeed the success obtained has been entirely dependent. The equation referred to is

$$d_t d_x u + d_y d_z u = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

By proceeding with this equation as in Art. V. we easily obtain the four following equations:

$$(d_td_x+d_yd_z)\,.\,td_tu+d_yd_zu=0\ldots\ldots\ldots\ldots\ldots(2),$$

$$(d_td_x+d_yd_z)\,.\,xd_xu+d_yd_zu=0\ldots\ldots\ldots\ldots\ldots(3),$$

$$(d_td_x+d_yd_z)\,.\,yd_yu+d_td_xu=0\ldots\ldots\ldots\ldots\ldots(4),$$

$$(d_td_x+d_yd_z)\,.\,zd_zu+d_td_xu=0\ldots\ldots\ldots\ldots\ldots(5).$$

Taking account of equation (1) we deduce from these the following results:

(2) − (3) gives $(d_td_x+d_yd_z)\,(td_tu-xd_xu)=0$,

(4) − (5) gives $(d_td_x+d_yd_z)\,(yd_yu-zd_zu)=0$,

(2) + (4) gives $(d_td_x+d_yd_z)\,(td_tu+yd_yu)=0$,

(2) + (5) gives $(d_td_x+d_yd_z)\,(td_tu+zd_zu)=0$,

(3) + (4) gives $(d_td_x+d_yd_z)\,(xd_xu+yd_yu)=0$,

(3) + (5) gives $(d_td_x+d_yd_z)\,(xd_xu+zd_zu)=0$.

It will be seen on examination that of the last six equations only *three* are independent. Integrating them with $(d_td_x+d_yd_z)^{-1}$, we obtain the following six first integrals of which only three are independent:

$$td_tu-xd_xu=k_1u,\ \text{and}\ yd_yu-zd_zu=k_2u\ \ldots\ldots(6),$$

$$td_tu+yd_yu=l_1u,\ \text{and}\ xd_xu+zd_zu=l_2u\ \ldots\ldots(7),$$

$$td_tu+zd_zu=m_1u,\ \text{and}\ xd_xu+yd_yu=m_2u\ldots\ldots(8).$$

The integrals of these are (taking them two and two as they stand),

$$u = t^{k_1} y^{k_2} F(tx,\ yz);$$

$$u = t^{l_1} x^{l_2} f\left(\frac{y}{t},\ \frac{z}{x}\right);$$

$$u = t^{m_1} x^{m_2} \phi\left(\frac{z}{t},\ \frac{y}{x}\right).$$

The forms of F, f, ϕ are to be determined by the separate substitution of these values of u in equation (1).

In this way, for example, we obtain from the first of these the following finite integral of the proposed equation,

$$Vu = F(\sqrt{tx} + \sqrt{-yz}) + f(\sqrt{tx} - \sqrt{-yz}),$$

the quantity represented by V being any one of these four following:

$$\sqrt{ty},\quad \sqrt{tz},\quad \sqrt{xy},\quad \sqrt{xz}.$$

The Theory of the Peculiar First Integrals.

(1) On turning to the three sets of first integrals (6), (7), (8), we perceive at once that no two of the sets can be satisfied with the same value of u. From this it follows that they are not to be used simultaneously. They point to three independent integrals of the proposed equation.

(2) On looking at the equations from which the *first* integrals were obtained we learn that the symbols

$$td_t - xd_x,\quad yd_y - zd_z,\quad xd_x + zd_z,\ \text{\&c.}$$

are commutative with $d_t d_x + d_y d_z$ on u.

(3) To avoid an unnecessary complexity of symbols let us confine ourselves to the consideration of the peculiar first integral in connexion with the equation

$$d_t d_x u + cu = 0 \dots\dots\dots\dots\dots\dots\dots\dots (9).$$

In this case $(d_t d_x + c)$ is commutative with $(td_t - xd_x)$ on u.

Let H represent any simple or compound symbol of operation which is thus commutative with $(d_t d_x + c)$ on u,

$$\therefore\ (d_t d_x + c)\, H \,.\, u = 0,$$

Integrating this equation with $(d_t d_x + c)^{-1}$ we have the following first integral,

$$H \,.\, u = ku \dots\dots\dots\dots\dots\dots\dots\dots (10).$$

If then we can find a form of H which will satisfy the one condition contained in its definition, viz. that it is commutative with $(d_t d_x + c)$ on u, the above first integral will hold good; and between (9) and (10) the final integral is to be obtained. Thus there will be as many first integrals as we can find independent forms for H; and as many final integrals as can be formed out of them.

(4) That H *may* have several values seems to be rendered probable by the fact that in the case of the equation we are now considering we already know two, viz.

$$H = d_t + md_x, \text{ and } H = td_t - xd_x \dots\dots\dots (11),$$

for both of these are of the first order and commutative with $(d_t d_x + c)$ on u.

The latter has been already considered; and the former gives, by the elimination of d_x from $d_t d_x u + cu = 0$, the following:

$$d_t^2 u - kd_t u - mcu = 0,$$

the integral of which is of the form

$$u = \epsilon^{\alpha t} F(x) + \epsilon^{\beta t} f(x).$$

And as t and x enter similarly into the proposed equation, this may be written in the form

$$u = A\epsilon^{\alpha t - \alpha' x} + B\epsilon^{\beta t - \beta' x},$$

in which $\alpha\alpha' = c$, and $\beta\beta' = c$. The two terms may be both included in the form

$$u = A\epsilon^{\alpha t - \alpha' x} + B\epsilon^{\alpha' x - \alpha t}.$$

Thus the existence of this final integral is accounted for by the fact that $d_t + md_x$ is commutative with $(d_t d_x + c)$.

(5) We have found two different and apparently independent forms of H, and from them two corresponding apparently independent integrals of the proposed equation. If these integrals are really independent, and even if the two forms of H are really independent, great uncertainty would exist as to the generality of the integral of

$$d_t d_x u + cu = 0$$

found in Art. V. We shall therefore now shew that the form of H given by the equation

$$H = td_t - xd_x \text{......................} (12)$$

includes the other form of H which we have just discussed.

We have not considered it necessary to remark respecting the equation

$$d_t d_x u + cu = 0,$$

and the other equations derived from it and discussed in Arts. IX., X., that as the independent variables t, x enter this equation only in the form dt, dx, we may write $t+c_1$, $x+c_2$ instead of t and x. This property is general for all independent variables which enter an equation only in the form of differentials. We ought therefore to have written in the integrals derived from those equations $t+c_1$, $x+c_2$, $y+c_2, \dots$ for $t, x, y, \dots$ We abstained from doing so for simplicity of symbolic expression.

Now if in (12) we write $at+l$, $bx+m$ for t and x, we have

$$H=\left(t+\frac{l}{a}\right)d_t-\left(x+\frac{m}{b}\right)d_x \dots\dots\dots\dots\dots (13),$$

and we observe that this form of H includes *both* the forms of H before given; and that H is precisely the same as if we had merely written

$$t+\frac{l}{a} \text{ for } t, \text{ and } x+\frac{m}{b} \text{ for } x.$$

We have therefore to make these changes in the integral of $d_t d_x u + cu = 0$ given in Art. v., and we shall then have the integral in a form that includes both forms of H. It is therefore not correct to say that H has *two* independent forms. The form discussed in section (4) of this Article is *necessarily* included in the form $td_t - xd_x$ by the general property that the general value of t is $t+c_1$, and that of x is $x+c_2$; in other words, the forms in which the independent variables enter into the proposed differential equations are such that we are at liberty arbitrarily to change the origins of the independent variables in the integrals.

(6) But can we reverse the process and shew that the integral

$$u = A\epsilon^{at-a'x} + B\epsilon^{a'x-at}.$$

is included in the general integral found in Art. v. the general form of which is

$$u\sqrt{t} = A\epsilon^{\sqrt{-ctx}} + B\epsilon^{-\sqrt{-ctx}},$$

by simply changing the origins of t and x? It can be done as follows.

For t and x write $t+\frac{l}{a}$ and $x+\frac{m}{a}$ in this integral; and for brevity let us confine ourselves to the first term of the integral, as the result for the other term can be deduced from it;

$$\therefore\ u\sqrt{t+\frac{l}{a}} = A\epsilon^{\left(t+\frac{l}{a}\right)^{\frac{1}{2}}\left(x+\frac{m}{a}\right)^{\frac{1}{2}}\sqrt{-c}};$$

$$\therefore\ u\sqrt{at+l} = A\sqrt{a}\,.\,\epsilon^{\frac{\sqrt{-clm}}{a}}\,.\,\epsilon^{\sqrt{-clm}\left(\frac{t}{2l}+\frac{x}{2m}+aW\right)};$$

wherein aW represents the remainder of the series obtained by the extraction of the root of

$$\left(1+\frac{at}{l}\right)\left(1+\frac{ax}{m}\right).$$

It is evident we may suppose the factors

$$\sqrt{a},\quad \epsilon^{\frac{\sqrt{-clm}}{a}},$$

which are independent of t and x, absorbed into the arbitrary constant, for which we may therefore write $A\sqrt{l}$;

$$\therefore\ u\sqrt{at+l} = A\sqrt{l}\,.\,\epsilon^{\sqrt{-clm}\left(\frac{t}{2l}+\frac{x}{2m}+aW\right)}.$$

Now as an extreme case let us suppose $a = 0$;

$$\therefore\ u = A\epsilon^{\sqrt{-clm}\,.\,\left(\frac{t}{2l}+\frac{x}{2m}\right)}.$$

The generality of this form is not diminished by the supposition that $lm+1=0$; and as we are at liberty to assume that l is negative and m positive, or conversely, that l is positive and m negative, we have the two following results corresponding to the two terms of the general integral,

$$u = A\epsilon^{\left(t\sqrt{\frac{m}{l}}-x\sqrt{\frac{l}{m}}\right)\sqrt{\frac{c}{2}}} + B\epsilon^{\left(x\sqrt{\frac{l}{m}}-t\sqrt{\frac{m}{l}}\right)\sqrt{\frac{c}{2}}}.$$

But it will be objected to this result that it has been obtained by assuming $t+\infty$ and $x+\infty$ for t and x; and that these assumptions are not lawful. We allow the illegality of such assumptions; and our reply is that these are not the assumptions we have made. If the reader will look back he will see that we assumed $t = at + l$, and $x = ax + m$; and that by these we were directed to write $t+\frac{l}{a}$ for t, and $x+\frac{m}{a}$ for x in *the integral*. We have done this, and it has conducted us to the above result.

We feel ourselves therefore entitled to affirm that the general value of H is $td_t - xd_x$; and that the general finite integral of the equation $d_t d_x u + cu = 0$ is that given in Art. v., with the understanding that we are at liberty to arbitrarily change the origins of. the independent variables in that integral.

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